Solution to a Diffusion problem by Finite Element Classic Method

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December 2014

Contents

Motivation

If we have a PDE, with certain boundary and initial conditions to be solved, the options we have are:

- To find an analytical solution u. This is the best, but not always available.
- To approximate solutions by u_h. We have always an error called the residual R.

Methods for solving approximately a PDE

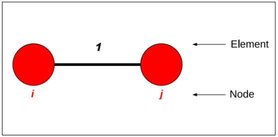
- Ritz Method.
- Variational Method.
- Least.Squares Method.
- Galerkin Method.
- Finite Difference Method.
- Finite Element Method.
- Finite Volume Method.

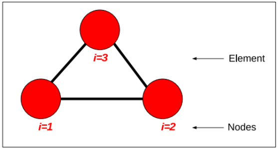
Basic steps I

In all FEM variants there are the same steps to be taken:

Discretize the continuum

- Divide the domain in smaller subregions called **elements**.
- The elements contains a certain number of points inside, called **nodes**.
- There are lots of shapes the elements can have (line segments, triangles, squares, etc.).
- For 1D the most simple shapes to consider are linear segments with two nodes per element. For 2D, the most simple are triangular elements.





Basic steps II

Determine the space of test (or weight) functions

- Select the kind of functions to take to describe the variation of the function u inside each element.
- This set of functions will describe the approximate solution u_h.
- One of the usual choices is to take linear polynomials ($\varphi(x) = a_0 + a_1 x$), or quadratic polynomials ($\varphi(x) = a_0 + a_1 x + a_2 x^2$).
- The value of the test function λ for a given position x inside an element can be written as a function of the values of φ at the N nodes of the element.

Basic steps III

Formulate the discrete problem

- For each element e we must find a system of algebraic equations such that, by solving it, we got the values of φ at the position of the nodes of e.
- For each element e we must find the matrix $[\mathbf{K}]_e$ and the vector $[\mathbf{f}]_\mathbf{e}$ such that

$$[\mathbf{K}]_e \cdot [u_h]_e = [\mathbf{f}]_e,$$

where $[u_h]_e = [u_1, u_2, \ldots, u_N]$ is the vector of values of u_h on the N nodes of e.

Basic steps IV

Assemble the equations for all the elements

- We have to assemble the equations for all the elements taking in account that, usually, contiguous elements have nodes in common, and two equations may refer to the same node.
- If we have a total of M effective nodes in the system, we have to build a global matrix $[\mathbf{K}]$ of size $M \times M$ and a global vector $[\mathbf{f}]$ of size M such that it reduces to solve the matrix equation

$$[\mathbf{K}] \cdot [u_h] = [\mathbf{f}],$$

where $[u_h] = [u_1, u_2, \ldots, u_M]$ is the vector of approximate values of u_h on each effective node.

Basic steps V

Solve the system of equations

- We are allowed to use whatever desired method to solve the system, but the more number of nodes, better is the quality of the approximate solution.
- The matrices we build up can be very large. The most of their entries are zero. So, sparse matrices structures are used to save a lot of time and memory.

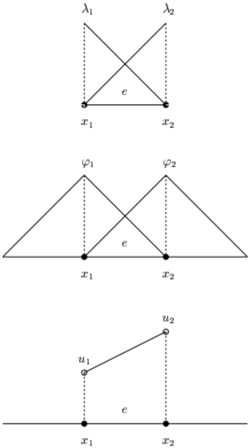
Linear Finite Elements

- Consider barycentric coordinates

$$\lambda_1(x) = \frac{x_2 - x}{x_2 - x_1}, \ \lambda_2(x) = \frac{x - x_1}{x_2 - x_1}$$

defined on the element $\mathbf{e} = [x_1, x_2]$.

- $\lambda_i \in P_1(\mathbf{e}), \ i = 1, 2.$
- $\lambda_i(x_j) = \delta_{ij}, \ i, j = 1, 2.$
- $\lambda_1(x) + \lambda_2(x) = 1, \ \forall x \in \mathbf{e}.$
- $u_h(x) = u_1\varphi_1(x) + u_2\varphi_2(x), \ \forall x \in e,$ where $\varphi_1|_e = \lambda_1, \ \varphi_2|_e = \lambda_2$ are the basis functions on e.



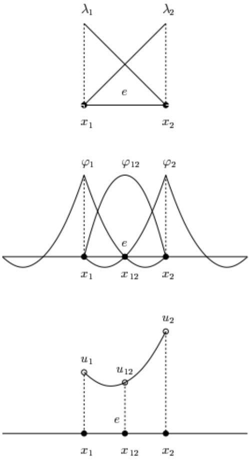
Quadratic Finite Elements

- Consider barycentric coordinates

$$\{\lambda_1(x), \lambda_2(x)\},$$

with the endpoints $x_1 = \{1, 0\}$, $x_2 = \{0, 1\}$ and midpoint $x_{12} = \frac{x_1+x_2}{2} = \{\frac{1}{2}, \frac{1}{2}\}$.

- Basis functions are $\varphi_1, \varphi_2, \varphi_{12} \in P_2(e)$
  $\varphi_1(x) = \lambda_1(x)(2\lambda_1(x) - 1)$
  $\varphi_2(x) = \lambda_2(x)(2\lambda_2(x) - 1)$
  $\varphi_{12}(x) = 4\lambda_1(x)\lambda_2(x)$
- $u_h|_e(x) =$
  $u_1\varphi_1(x) + u_2\varphi_2(x) + u_{12}\varphi_{12}(x).$



Poisson equation (Stationary case of Diffusion equation)

- We consider the elliptic problem

$$-\frac{d}{dx}\left(D\frac{du}{dx}\right) = f \text{ in } (0,1)$$

$$u(0) = 0, \;\; u(1) = 0.$$

- $u_h = \sum_{j=1}^{M} u_j \varphi_j$.
- We discretise on the space domain [0,1], multiplying both sides of the PDE by a test function φ and integrating by parts on each element e.
- Because $\varphi(0) = 0 = \varphi(1)$ we have

$$u_0 = 0, \;\; \sum_{j=1}^{M} u_j \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} dx = \int_0^1 f\varphi_i dx, \;\; \forall i = 1, 2, \ldots, M.$$

Linear system of equations I

- Decomposite integrals into elements contributions $e_k = [x_{k-1}, x_k]$,

$$\sum_{j=1}^{M} u_j \sum_{k=1}^{M} \int_{e_k} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} dx = \sum_{k=1}^{M} \int_{e_k} f\varphi_i dx, \ \forall i = 1, 2, \ldots, M.$$

- The global system has the form $Au_h = F$, where

$$a_{ij} = \sum_{k=1}^{M} a_{ij}^k = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} dx \ \ a_{ij}^k \neq 0 \text{ only for } i, j \in \{k-1, k\}$$

$$F_i = \sum_{k=1}^{M} F_i^k = \int_0^1 f\varphi_i dx \ \ F_i^k \neq 0 \text{ only for } i \in \{k-1, k\}$$

Linear system of equations II

- The element blocks for $e_k = [x_{k-1}, x_k]$ in the stiffness matrix and the load vector are

$$A^k = \begin{bmatrix} \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_{k-1}}{dx}dx & \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_{k}}{dx}dx \\ \int_{e_k} \frac{d\varphi_{k}}{dx}\frac{d\varphi_{k-1}}{dx}dx & \int_{e_k} \frac{d\varphi_{k}}{dx}\frac{d\varphi_{k}}{dx}dx \end{bmatrix}, \; F^k = \begin{bmatrix} \int_{e_k} f\varphi_{k-1}dx \\ \int_{e_k} f\varphi_k dx \end{bmatrix}.$$

- Coefficients of the global system $Au_h = F$ are

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{k-1,k-1} & a_{k-1,k} & \cdot \\ \cdot & a_{k,k-1} & a_{k,k} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \; F = \begin{bmatrix} \cdot \\ F_{k-1} \\ F_k \\ \cdot \end{bmatrix}$$

Numerical integration

- Change of variable: $\int_e f(x)dx = \int_{\widehat{e}} \widehat{f}(\widehat{x})|detJ|d\widehat{x}$, where $\varphi_i(x) = \widehat{\varphi}_i(F_e^{-1}(x))$, $\forall x \in e$.
- $\frac{d\widehat{\varphi}_i}{d\widehat{x}} = J\frac{d\varphi_i}{dx}$ implies $\frac{d\varphi_i}{dx} = J^{-1}\frac{d\widehat{\varphi}_i}{d\widehat{x}}$.
- The entries of the stiffness matrix are given by

$$a_{ij} = \int_e \frac{d\varphi_i}{dx}\frac{\varphi_j}{dx}dx = \int_{\widehat{e}} \left(J^{-1}\frac{d\widehat{\varphi}_i}{d\widehat{x}}\right)\left(J^{-1}\frac{d\widehat{\varphi}_j}{d\widehat{x}}\right)|detJ|d\widehat{x}$$

- Numerical integration formulae are used to compute the integrals

$$\int_{\widehat{e}} \widehat{g}(\widehat{x})d\widehat{x} \approx \sum_{i=0}^{n} \widehat{w}_i\widehat{g}(\widehat{x}_i),\ \widehat{g}(\widehat{x}) = \widehat{f}(\widehat{x})|detJ|$$

- Gauss-Legendre quadratures are preferable over Newton-Cotes formulae.

Weak formulation of the evolution problem

- The goal is now to numerically solve the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) = f \text{ in } (0,1)\times[0,1],$$

$$u(0,t) = 0,\ \ u(1,t) = 0,$$

$$u(x,0) = u^0(x) = x.$$

- The weak form is

$$\sum_{j=1}^{M}\left(u_j^n - u_j^{n-1}\right)\int_0^1 \varphi_j \frac{\partial \varphi}{\partial t} dx + \sum_{j=1}^{M} u_j^n \int_0^1 \frac{\partial \varphi_i}{\partial x}\frac{\partial \varphi_j}{\partial x} dx = \int_0^1 f^n \varphi_i dx,$$

$$\forall i = 1,2,\ldots,M,\ \forall n = 1,2,\ldots,N.$$

Discrete form of evolution equation

- The discrete form of diffusion equation is

$$M + \Delta t A) u_j^n = u_j^{n-1} + \Delta t F,$$

  where $M = |detJ|^{-1}[\int_{\widehat{e}} \widehat{\varphi}_i \widehat{\varphi}_j]$.
- To solve for time t_n being known the solution for time t_{n-1} we use Backward Euler Method.
- To solve the linear system in each step of Backward Euler Method, we can use any iterative method (like Conjugate Gradient) or direct method (like Back substitution).

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