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On the Well-Posedness of First-Order Variable Exponent Cauchy Problems with Robin and Wentzell-Robin Boundary Conditions on Arbitrary Domains

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Abstract. We define the notion of relative capacity of variable exponent type, referred in this article as the relative $p(\cdot)$-capacity, and use this approach to obtain a necessary and sufficient condition for the well-posedness of the corresponding parabolic boundary value problems involving the $p(\cdot)$-Laplace operator and either Robin or Wentzell-Robin boundary conditions on arbitrary domains.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) be an arbitrary domain with finite Lebesgue measure, and with boundary $\partial\Omega$, let μ be a finite Borel regular measure supported on $\partial\Omega$, and let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 \leq p_* := \operatorname{ess}\inf_{\overline{\Omega}} p(x) \leq p^* := \operatorname{ess}\sup_{\overline{\Omega}} p(x) < \infty$ (see section 2 for the definition of this space). The aim of this paper is to investigate the well-posedness of the first order Cauchy problem involving the $p(\cdot)$-Laplace operator and either Robin or Wentzell boundary conditions on general domains. If Ω is "sufficiently regular", for instance, a Lipschitz domain, then the Robin boundary conditions of our problem of interest are given by

$$|\nabla u|^{p(\cdot)-2}\frac{\partial u}{\partial \nu} + \beta|u|^{p(\cdot)-2}u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

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while the Wentzell, or so called Wentzell-Robin boundary conditions are defined by

$$\Delta_{p(\cdot)}u + |\nabla u|^{p(\cdot)-2}\frac{\partial u}{\partial \nu} + \beta|u|^{p(\cdot)-2}u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

for $\beta \in L^{\infty}(\partial\Omega, d\mu)$ with $\inf_{x\in\partial\Omega}\beta(x) \geq \beta_0$ for some constant $\beta_0 > 0$, where $\Delta_{p(\cdot)}u := \text{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$ denotes the $p(\cdot)$-Laplace operators in $\partial\Omega$. However, if Ω is "sufficiently bad", then the above boundary conditions (1.1) and (1.2) may not make sense, and in particular it is known that the classical normal derivative may not exist in such domains. On the other hand, in this paper, by using a capacity approach, we will characterize the class of finite measures on $\partial\Omega$ where the parabolic equations involving the $p(\cdot)$-Laplace operator and generalized boundary conditions of the form (1.1) and (1.2) are well-defined. For this purpose, we will need to give sense to the notion of a generalized normal derivative related to the appropriate measure on the boundary $\partial\Omega$. This will be discussed in detail in section 4.

If $p \in (1,\infty)$ is constant, then the p-Laplace Eq. with Robin boundary conditions has been investigated by many authors on either smooth domains, and also on non-smooth domains. For $p = 2$, the well-posedness of the Robin boundary value problem on arbitrary domains was introduced by Daners [15] for bounded domains with boundary finite with respect to the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$, and generalized to arbitrary domains with respect to $\text{Cap}_{2,\Omega}$-admissible measures on the boundary (see Definition 3.11) by Arendt and Warma [4]. A realization of the Robin boundary value problem in fractal domains can be found in [5]. Generalizations of the above results to $p \in (1,\infty)$ can be found in [10, 37] on bounded $W^{1,p}$-extension domains (see Definition 2.3), and in [16, 41] on arbitrary domains. In particular, it was shown in [41] that the first order Cauchy problem involving the p-Laplace operator and boundary conditions of the form (1.1) is well-posed on an arbitrary open set if and only if the measure $\mathcal{H}^{N-1}$ is $\text{Cap}_{p,\Omega}$-admissible. On the other hand, Eq. with Wentzell-Robin boundary conditions of type (1.2) have been well investigated for $p = 2$. A generalization to $p \in (1,\infty)$ was introduced by Warma [39], and investigated further by the same author in [40]. However, all the results obtained for the Wentzell-Robin problem assumed Ω to be at least a bounded Lipschitz domain. Our goal in this article is to generalize the result in [41] previously discussed above, to the variable exponent case, to include in addition other finite Borel regular measures on the boundary, and finally to obtain the same conclusions for parabolic Eq. with boundary conditions of the type (1.2) on arbitrary domains. This last fact has not been discussed much in the literature.

Over the last years, the study of variable exponent function spaces and differential equation have experienced a substantial growth, and have attracted a number of authors in several areas, motivated by various applications, such as electrorheological fluids, image restoration and modern engineering, among others (e.g. [1, 13, 19, 20, 34]). However, as one may expect, many boundary value problems of variable exponent type have not been investigated in full strength, mainly because it has become necessary to establish the validity of many properties, valid for the constant case, to the variable exponent case. Some properties may even fail for variable exponents, and thus the situation becomes much more delicate in several areas. Concerning the Robin boundary condition (1.1), it has been studied by several authors (e.g. [9, 18], among others), but the Wentzell-Robin boundary value problem with variable exponent has been unknown in our knowledge, and no literature has been found concerning this kind of differential equations, up to these last years (by the author).

The structure of this paper is as follows. In section 2 we give the framework on which the results of the subsequent sections will be obtained, and we introduce the notations, definitions, and well-known results that will be applied throughout the rest of the article. In section 3 we introduce the notion of the relative $p(\cdot)$-capacity as a generalization of the relative capacity for $p \in [1,\infty)$ defined by Biegert [7, 8], to the variable exponent case. We remark that there is a notion of relative capacity for variable exponents given in [21, Section 10.2] which differs from the one we were interested, since our definition will agree with the definition given in [7, 8] when p is constant. Then we will establish several properties of this capacity similar as in the constant case, and also will introduce the notion of $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible measures, which will play a key role in the paper. Finally, in section 4 we apply the results of the previous section to give a necessary and sufficient condition for the well-posedness of the $p(\cdot)$-Laplace Eq. with either Robin, or Wentzell-Robin boundary conditions on arbitrary domains. More precisely, we prove that the first order Cauchy problem involving the $p(\cdot)$-Laplace operator and boundary conditions of type (1.1) and (1.2) (in the generalized sense) is well-posed on $L^{q(\cdot)}(\Omega, dx)$ and $L^{q(\cdot)}(\Omega, dx) \times L^{q(\cdot)}(\partial\Omega, d\mu)$ for each measurable function $q(\cdot)$ over $\overline{\Omega}$ such that $1 \leq q_* \leq q^* < \infty$, respectively, if and only if the Borel regular measure μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible over $\partial\Omega$. Observe that the well-posedness of the Wentzell problem on arbitrary domains implies that, given the appropriate measure on $\partial\Omega$, one can give sense to the boundary operator $\Delta_{p(\cdot)}$ over $\partial\Omega$. At the end, several examples of domains and measures where the stated conclusions hold are given.

2 Preliminaries and intermediate results

Throughout this paper we let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a domain with finite measure whose boundary $\partial\Omega$ is finite with respect to a Borel regular measure μ, and we let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 \leq p_* := \operatorname{ess\,inf}_{\overline{\Omega}} p(x) \leq p^* := \operatorname{ess\,sup}_{\overline{\Omega}} p(x) < \infty$. Here $\mathcal{P}^{\log}(\overline{\Omega})$ denotes the set of functions $u \in \mathcal{P}(\overline{\Omega}) := \{p : \overline{\Omega} \to [1, \infty] \text{ measurable}\}$ such that the function $v := 1/u$ is globally log-Hölder continuous, that is, if there exist constants $c_1, c_2 > 0$ and a constant $\alpha \in \mathbb{R}$ such that

$$|v(x) - v(y)| \leq \frac{c_1}{\log(e + 1/|x-y|)} \qquad \text{and} \qquad |v(x) - \alpha| \leq \frac{c_2}{\log(e + |x|)}$$

for all $x, y \in \overline{\Omega}$. For properties of the space $\mathcal{P}^{\log}(\overline{\Omega})$, we refer to [21, Section 4.1].

Given $E \subseteq \overline{\Omega}$ a positive measure space with respect to a finite Borel measure ν, set $E^p_\infty := \{x \in E \mid p(x) = \infty\}$. We define

$$L^{p(\cdot)}(E, d\nu) := \left\{u : E \to [-\infty, \infty] \text{ measurable} \mid \rho_{p,E}(u) < \infty\right\},$$

where

$$\rho_{p,E}(u) := \int_{E \setminus E^p_\infty} |u(x)|^{p(x)}\, d\nu + \|u\|_{L^\infty(E^p_\infty, \nu)}.$$

Because of our assumptions on the function p, it is easy to see that in our case $E^p_\infty = \emptyset$ and $E \setminus E^p_\infty = E$, so $L^{p(\cdot)}(E, d\nu)$ becomes the Musielak-Orlicz space $L^{\varphi_p}(E, d\nu)$ for $\varphi_p(x, u) := |u|^{p(x)}$, endowed with the Luxemburg norm

$$\|u\|_{p(\cdot),E} := \|u\|_{L^{p(\cdot)}(E,d\nu)} := \inf\left\{\lambda > 0 \mid \rho_{p,E}(u/\lambda) \leq 1\right\}.$$

(e.g. [33, Theorems 1.6 and 7.7]). The variable exponent L^p spaces of our interest will be $L^{p(\cdot)}(\Omega, dx)$ and $L^{p(\cdot)}(\partial\Omega, d\mu)$.

We will also consider the first order Sobolev space with variable exponent, defined by

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega, dx) \mid \nabla u \in L^{p(\cdot)}(\Omega, dx)^N\right\},$$

and endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(u/\lambda) + \rho_{p,\Omega}(|\nabla u|/\lambda) \leq 1\right\}.$$

For the classical properties of the variable exponent Lebesgue and Sobolev spaces, refer to [21, 23, 29, 33]. Also, for $E \subseteq \overline{\Omega}$ and ν a Borel measure supported in E, we define the so called variable exponent Hajłasz-Sobolev space $M^{1,p(\cdot)}(E)$ as the set of functions $u \in L^{p(\cdot)}(E, d\nu)$ such that there exist a nonnegative function $v \in L^{p(\cdot)}(E, d\nu)$ fulfilling the inequality

$$|u(x) - u(y)| \leq |x - y|(v(x) + v(y)) \qquad \text{for } \nu\text{-almost all } x \in E.$$

The function v is called the Hajłasz gradient of u, and it is known that $M^{1,p(\cdot)}(E)$ is a Banach space with respect to the norm

$$\|u\|_{M^{1,p(\cdot)}(E)} := \|u\|_{p(\cdot),E} + \inf \|v\|_{p(\cdot),E}$$

(e.g. [24, 26]). For more properties of Hajłasz-Sobolev spaces, we refer to [3, 26].

Now we give some definitions that will be quoted later on. We begin by stating the definition of the generalized variable exponent Maz'ya space on an open set, given in [36] for bounded domains.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let μ be a finite Borel measure supported on $\partial\Omega$. Given $p, r \in \mathcal{P}^{\log}(\overline{\Omega})$ with $1 \leq p_* \leq p^* < \infty$ and $1 \leq r_* \leq r^* < \infty$, we define the *extended variable exponent Maz'ya space* $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu)$ to be the completion of the space

$$V^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) := \{u \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid u|_{\partial\Omega} \in L^{r(\cdot)}(\partial\Omega, d\mu)\}$$

with respect to the norm

$$\|u\|_{W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(|\nabla u|/\lambda) + \rho_{p,\Omega}(u/\lambda) + \rho_{r,\partial\Omega}(u/\lambda) \leq 1\right\}.$$

Definition 2.2. Let $d \in (0, N)$ and μ a Borel measure supported on a bounded set $F \subseteq \mathbb{R}^N$. Then μ is said to be an *upper d-Ahlfors measure*, if there exist constants $M, R_0 > 0$ such that

$$\mu(B_r(x)) \leq Mr^d, \qquad \text{for all } 0 < r < R_0 \text{ and } x \in F, \tag{2.1}$$

where $B_r(x)$ denotes the ball of radius r centered at $x \in F$. On the other hand, the measure μ as called a *lower d-Ahlfors measure*, if the reverse inequality in (2.1) is fulfilled.

If the condition (2.1) and its reverse inequality are both fulfilled, then the set $F \subseteq \mathbb{R}^N$ is called a *d-set* with respect to the measure μ (e.g. [12]) Moreover, the above condition can be reformulated as

$$\mu(B_r(x)) \leq M\frac{\lambda_N(B_r(x))}{r^{N-d}}, \qquad 0 < r < R_0 \text{ and } x \in F \tag{2.2}$$

(e.g. [17]), where $\lambda_N(\cdot)$ denotes the N-dimensional Lebesgue measure on $\mathbb{R}^N$.

Definition 2.3. Let $p \in \mathcal{P}^{\log}(\overline{\Omega})$, $1 \leq p_* \leq p^* < \infty$. An open set $\Omega \subseteq \mathbb{R}^N$ is called a $W^{1,p(\cdot)}$-*extension domain*, if Ω has the $W^{1,p(\cdot)}$-extension property, that is, if there exists a bounded linear operator $P : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^N)$ such that $Pu = u$ a.e. on Ω. If in addition we have $P(W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})) \subseteq W^{1,p(\cdot)}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, then we say that Ω has the *continuous* $W^{1,p(\cdot)}$*-extension property*.

Remark 2.4. If $\Omega \subseteq \mathbb{R}^N$ is a domain, and if $p \in \mathcal{P}^{\log}(\overline{\Omega})$, then it follows from [21, Proposition 4.1.7] that p can be extended to a function $\tilde{p} \in \mathcal{P}^{\log}(\mathbb{R}^N)$ with $\tilde{p}_* = p_*$, $\tilde{p}^* = p^*$, and with the same log-Hölder constant.

Remark 2.5. Let Ω be a bounded $W^{1,p(\cdot)}$-extension domain, let $p, r \in C^{0,1}(\overline{\Omega})$, that is, let p and r be Hölder continuous functions over $\overline{\Omega}$, satisfying $1 < p_* \le p^* < N$ and $1 < r_* \le r(x) \le dp(x)(N-p_*)^{-1}$, and let μ be an upper d-Ahlfors measure on $\partial\Omega$, for $d \in (N-p_*, N)$. Then it follows from [36, Corollary 4.7 and Theorem 4.9] that

$$W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) = W^{1,p(\cdot)}(\Omega) = W_\mu(\Omega, \partial\Omega),$$

up to equivalent norms, where $W_\mu(\Omega,\partial\Omega)$ is defined as the completion of the space

$$V_\mu(\Omega,\partial\Omega) := \{u \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid u|_{\partial\Omega} \in L^{r(\cdot)}(\partial\Omega, d\mu)\}$$

with respect to the norm

$$\|u\|_{W_\mu(\Omega,\partial\Omega)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(|\nabla u|/\lambda) + \rho_{r,\partial\Omega}(u/\lambda) \le 1\right\}.$$

Moreover, it was obtained in [36, Theorem 3.1] that in this case the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\Omega, dx)$ is continuous, and in fact the proof of this result shows that this conclusion is also valid for $p \in \mathcal{P}^{\log}(\overline{\Omega})$. Note that if $p \in (1,N)$ is constant and $\mu = \mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure supported on $\partial\Omega$, then $W_\mu(\Omega,\partial\Omega)$ agrees with the classical Maz'ya space introduced by Maz'ya [31] in the constant case.

Example 2.6. If $\Omega \subseteq \mathbb{R}^N$ is an (ϵ,δ)-domain (see [28] for the definition of this domain), then Ω is a $W^{1,p(\cdot)}$-extension domain (e.g. [21, 36]). In particular, if Ω is the classical snowflake domain, then by [28] it is an (ϵ,δ)-domain, and by [38] its boundary is a d-set with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$, where $d := \log(4)/\log(3)$. Another example of an extension domain whose fractal boundary is a d-set with respect to the so called *self-similar measure* (see [22] for the definition of this measure) can be found in [2]. Finally, it is well-known that Lipschitz domains are $W^{1,p(\cdot)}$-extension domains whose boundary is a $(N-1)$-set with respect to the classical Surface measure $\sigma = \mathcal{H}^{N-1}$.

Now we state some known results that will be applied in the subsequent sections. These results generalize some results in [25, 35] for the constant case.

Proposition 2.7. (see [26]) *If $p \in \mathcal{P}^{\log}(\mathbb{R}^N)$ with $1 \le p_* \le p^* < \infty$, then $M^{1,p(\cdot)}(\mathbb{R}^N) = W^{1,p(\cdot)}(\mathbb{R}^N)$.*

Theorem 2.8. (see [27]) *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, where $p \in \mathcal{P}^{\log}(\overline{\Omega})$ fulfills $1 < p_* \le p^* < \infty$. Then the measure density condition*

$$\lambda_N(B_r(x) \cap \Omega) \ge c\, r^N \tag{2.3}$$

holds some constant $c > 0$, and for all $x \in \Omega$ and $r \in (0,1]$, where we recall that λ_N denotes the N-dimensional Lebesgue measure on Ω.

When $\Omega \subseteq \mathbb{R}^N$ is bounded, it is obvious that the inequality $\lambda_N(B_r(x)\cap\Omega) \le c' r^N$ is valid for some constant $c' > 0$ and for every $x \in \Omega$ and $r > 0$. Thus by Theorem 2.8 one sees that if in addition Ω

is a bounded $W^{1,p(\cdot)}$-extension domain for $p \in \mathcal{P}^{\log}(\overline{\Omega})$ with $1 < p_* \le p^* < \infty$, then Ω is a N-set with respect to the Lebesgue measure λ_N.

Theorem 2.9. (see [27]) *Let $E \subseteq \overline{\Omega}$ be a measurable set satisfying the measure density condition (2.3), and let $p \in \mathcal{P}^{\log}(E)$ be such that $1 < p_* \le p^* < \infty$. Then $M^{1,p(\cdot)}(\mathbb{R}^N)|_E = M^{1,p(\cdot)}(E)$, and there exists a bounded extension operator $\varepsilon : M^{1,p(\cdot)}(E) \to M^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\varepsilon(u) = u$ a.e. in E.*

To conclude this section we give recall briefly some facts about nonlinear semigroups. Indeed, let H be a Hilbert space, and let $\varphi : H \to (-\infty, \infty]$ be a proper, convex, lower semicontinuous functional with effective domain $D(\varphi) := \{u \in H \mid \varphi(u) < \infty\}$. Clearly $D(\varphi) \subseteq H$ is convex. Then the *subdifferential* $\partial\varphi$ of φ is defined by

$$\begin{cases} D(\partial\varphi) := \{u \in D(\varphi) \mid \exists w \in H,\ \varphi(v) - \varphi(u) \ge \langle w, v-u \rangle_H, \text{ for all } v \in H\}, \\ \partial\varphi(u) := \{w \in H \mid \varphi(v) - \varphi(u) \ge \langle w, v-u \rangle_H, \text{ for all } v \in H\}, \end{cases}$$

where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on H. We close this section with the following classical result.

Theorem 2.10. (see [32]) *The subdifferential $\partial\varphi$ is a maximal monotone operator. Moreover, $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$. The subdifferential $\partial\varphi$ generates a (nonlinear) C_0-semigroup $\{T(t)\}_{t\ge0}$ on $\overline{D(\varphi)}$ in the following sense: for each $u_0 \in \overline{D(\varphi)}$, the function $u := T(\cdot)u_0$ is the unique strong solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{loc}((0,\infty); H) \text{ and } u(t) \in \partial\varphi \text{ a.e.}, \\ \dfrac{\partial u}{\partial t} + \partial\varphi(u) = 0 \text{ a.e. on } \mathbb{R}_+, \\ u(0,x) = u_0(x). \end{cases}$$

In addition, the subdifferential $\partial\varphi$ generates a (nonlinear) semigroup $\{\tilde{T}(t)\}_{t\ge0}$ on H, where for every $t \ge 0$, $\tilde{T}(t)$ is the composition of the semigroup $T(t)$ on $\overline{D(\varphi)}$ with the projection on the convex set $\overline{D(\varphi)}$.

Definition 2.11. Let $\{T(t)\}_{t\ge0}$ be a (nonlinear) semigroup on a Hilbert lattice H with ordering $\le$, let X be a locally compact metric space, and ν a Borel regular measure on X.

(a) $\{T(t)\}_{t\ge0}$ is said to be *order-preserving*, if

$$T(t)u \le T(t)v \text{ for all } t \ge 0, \text{ whenever } u, v \in H,\ u \le v.$$

(b) $\{T(t)\}_{t\ge0}$ is said to be *submarkovian*, if

$$\|T(t)u - T(t)v\|_{\infty,X} \le \|u - v\|_{\infty,X}, \text{ for every } t \ge 0 \text{ and } u, v \in L^2(X, d\nu) \cap L^\infty(X, d\nu).$$

The next two well-known results characterize the order-preserve property and the submarkovian property of the functional φ, respectively.

Proposition 2.12. (see [14]) *Let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional on a real Hilbert lattice H, with effective domain $D(\varphi)$. Let $\{T(t)\}_{t\ge0}$ be the (nonlinear) semigroup on H generated by $\partial\varphi$. Then the following assertions are equivalent.*

(i) *The semigroup $\{T(t)\}_{t\ge0}$ is order preserving.*

(ii) *For all* $u, v \in H$ *one has*

$$\varphi\left(\frac{1}{2}(u + u \wedge v)\right) + \varphi\left(\frac{1}{2}(v + u \vee v)\right) \leq \varphi(u) + \varphi(v),$$

where $u \wedge v := \inf\{u, v\}$ and $u \vee v := \sup\{u, v\}$.

Proposition 2.13. (see [14]) *Let* $\varphi : L^2(X, d\nu) \to (-\infty, +\infty]$ *be a proper, convex, lower semicontinuous functional. Let* $\{T(t)\}_{t\geq 0}$ *be the (nonlinear) semigroup on* $L^2(X, d\nu)$ *generated by* $\partial\varphi$. *Assume that* $\{T(t)\}_{t\geq 0}$ *is order preserving. Then, the following assertions are equivalent.*

(i) *The semigroup* $\{T(t)\}_{t\geq 0}$ *is submarkovian.*

(ii) *For all* $u, v \in L^2(X, d\nu)$ *and* $\alpha > 0$,

$$\varphi\Big(v + g_\alpha(u, v)\Big) + \varphi\Big(u - g_\alpha(u, v)\Big) \leq \varphi(u) + \varphi(v),$$

where

$$g_\alpha(u, v) := \frac{1}{2}\Big[(u - v + \alpha)^+ - (u - v - \alpha)^-\Big],$$

with $u^+ := \sup\{u, 0\}$, *and* $u^- := \sup\{-u, 0\}$.

3 The relative $p(\cdot)$-capacity

In this section we present the theory of the relative $p(\cdot)$-capacity. This will be the key property that will be fundamental in the main results. We begin with the following definitions.

Definition 3.1. Let $E \subseteq \mathbb{R}^N$ and let $p \in \mathcal{P}(\mathbb{R}^N)$ be such that $1 \leq p_* \leq p^* < \infty$. The $p(\cdot)$-*capacity* of E is defined by

$$\mathrm{Cap}_{p(\cdot)}(E) := \inf_{u \in S_{p(\cdot)}(E)} \Big\{\rho_{p,\mathbb{R}^N}(u) + \rho_{p,\mathbb{R}^N}(|\nabla u|)\Big\},$$

where

$$S_{p(\cdot)}(E) := \Big\{u \in W^{1,p(\cdot)}(\mathbb{R}^N) \mid u \geq 0 \text{ and } u \geq 1 \text{ in an open set containing } E\Big\}.$$

Note that if $p \in [1, \infty)$ is constant, then the above definition agree with the well-know definition of the classical p-capacity. Moreover, if $S_{p(\cdot)}(E) = \emptyset$, then one has $\mathrm{Cap}_{p(\cdot)}(E) = 0$. For more properties of $\mathrm{Cap}_{p(\cdot)}(\cdot)$, we refer to [21, Section 10.1].

Definition 3.2. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $p \in \mathcal{P}(\overline{\Omega})$ be such that $1 \leq p_* \leq p^* < \infty$, let $E \subseteq \overline{\Omega}$, and let $\widetilde{W}^{1,p(\cdot)}(\Omega)$ denote the closure in $W^{1,p(\cdot)}(\Omega)$ of the space $W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega})$. The *relative* $p(\cdot)$-*capacity* of E with respect to Ω is defined by

$$\mathrm{Cap}_{p(\cdot),\Omega}(E) := \inf_{u \in S_{p(\cdot),\Omega}(E)} \Big\{\rho_{p,\Omega}(u) + \rho_{p,\Omega}(|\nabla u|)\Big\},$$

where

$$S_{p(\cdot),\Omega}(E) := \left\{u \in \widetilde{W}^{1,p(\cdot)}(\Omega) \mid \exists\ U \subseteq \mathbb{R}^N \text{ open, } E \subseteq U \text{ and } u \geq 1 \text{ a.e. on } \Omega \cap U\right\}.$$

Observe that if $\Omega = \mathbb{R}^N$, then $\mathrm{Cap}_{p(\cdot),\Omega}(E) = \mathrm{Cap}_{p(\cdot)}(E)$ for each $E \subseteq \mathbb{R}^N$.

Definition 3.3. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $p \in \mathcal{P}(\overline{\Omega})$ be such that $1 \leq p_* \leq p^* < \infty$, and let $E \subseteq \overline{\Omega}$.

(1) E is said to be $\mathrm{Cap}_{p(\cdot),\Omega}$-*polar*, if $\mathrm{Cap}_{p(\cdot),\Omega}(E) = 0$.

(2) We say that a property holds $\mathrm{Cap}_{p(\cdot),\Omega}$-*quasi everywhere* (abbreviated $p(\cdot)$-q.e.) on E, if there exists a $\mathrm{Cap}_{p(\cdot),\Omega}$-polar set D such that the property holds over $E \setminus D$.

(3) A function u is called $\mathrm{Cap}_{p(\cdot),\Omega}$-*quasi continuous* on E, if for all $\epsilon > 0$, there exists a open set $U \subseteq \overline{\Omega}$ such that $\mathrm{Cap}_{p(\cdot),\Omega}(U) \leq \epsilon$ and $u|_{E \setminus U}$ is continuous.

The notion of $p(\cdot)$-relative capacity was first introduced by Arendt and Warma [4], and generalized to $p \in [1, \infty)$ and to modular spaces by Biegert [6, 7, 8, 10]. We give below some properties which follow as consequences of the results of Biegert [6], valid for modular spaces (see also [8] for the constant case).

Proposition 3.4. (see [6]) *The relative $p(\cdot)$-capacity is a normed Choquet capacity on $\overline{\Omega}$, and for every $E \subseteq \overline{\Omega}$ we have that*

$$\mathrm{Cap}_{p(\cdot),\Omega}(E) = \inf\left\{\mathrm{Cap}_{p(\cdot),\Omega}(U) \mid U \subseteq \overline{\Omega} \text{ open, and } E \subseteq U\right\} \tag{3.1}$$

Proposition 3.5. (see [6]) *If $E \subseteq \overline{\Omega}$ is $\mathrm{Cap}_{p(\cdot),\Omega}$-polar, then $\lambda_N(E) = 0$.*

Proposition 3.6. (see [6]) *For every $u \in \widetilde{W}^{1,p(\cdot)}(\Omega)$ there exists a $\mathrm{Cap}_{p(\cdot),\Omega}$-quasi continuous function $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\tilde{u} = u$ a.e. on Ω.*

Proposition 3.7. (see [6]) *If $K \subseteq \overline{\Omega}$ is compact, then*

$$\begin{aligned}\mathrm{Cap}_{p(\cdot),\Omega}(K) &= \inf\left\{\rho_{p,\Omega}(u) + \rho_{p,\Omega}(|\nabla u|) \mid u \in \widetilde{W}^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}),\ u \geq 1 \text{ on } K\right\} \\ &= \inf\left\{\rho_{p,\Omega}(u) + \rho_{p,\Omega}(|\nabla u|) \mid u \in \widetilde{W}^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega}),\ u \geq 1 \text{ on } K\right\}.\end{aligned}$$

Now we are interested in giving a key relation between the $p(\cdot)$-capacity and the relative $p(\cdot)$-capacity for a class of bounded domains. To do so, we begin with the following result that was obtained for the constant case in [8, Theorem 3.13].

Lemma 3.8. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, and let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < \infty$. Then Ω has the continuous $W^{1,p(\cdot)}$-extension property.*

Proof: By virtue of Theorem 2.8 we see that the measure density condition (2.3) holds for some constant $c > 0$ and for all $x \in \Omega$ and $r \in (0, 1]$, and moreover one has $\lambda_N(\partial\Omega) = 0$. In addition, by applying Theorem 2.9 we deduce that $M^{1,p(\cdot)}(\mathbb{R}^N)|_{\overline{\Omega}} = M^{1,p(\cdot)}(\overline{\Omega})$, and there exists a bounded extension operator $\varepsilon_p : M^{1,p(\cdot)}(\overline{\Omega}) \to M^{1,p(\cdot)}(\mathbb{R}^N)$. Because $p \in \mathcal{P}^{\log}(\overline{\Omega})$, by Proposition 2.7 we have $M^{1,p(\cdot)}(\mathbb{R}^N) = W^{1,p(\cdot)}(\mathbb{R}^N)$, and thus $M^{1,p(\cdot)}(\overline{\Omega}) = W^{1,p(\cdot)}(\Omega)$, up to equivalent norms, and all this imply that the operator ε_p is also a continuous extension operator from $W^{1,p(\cdot)}(\Omega)$ onto $W^{1,p(\cdot)}(\mathbb{R}^N)$. Finally, to show that the extension operator ε_p maps $W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ into $W^{1,p(\cdot)}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, we just examine the construction of the operator ε_p given in [27, Theorem 3.4], and this completes the proof. □

Theorem 3.9. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, and let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \le p^* < \infty$. Then there exists a constant $c_\Omega = c(\Omega)$ such that*

$$\mathrm{Cap}_{p(\cdot)}(E) \le c_\Omega \mathrm{Cap}_{p(\cdot),\Omega}(E)^{\hat{p}/\bar{p}} \le c_\Omega \mathrm{Cap}_{p(\cdot)}(E)^{\hat{p}/\bar{p}} \tag{3.2}$$

for every set $E \subseteq \overline{\Omega}$, where $\hat{p}$ and $\bar{p}$ are positive constants that will be specified in the proof.

Proof: We first prove the inequality at the right hand side of (3.2). In fact, let $u \in S_{p(\cdot),\mathbb{R}^N}(E) = S_{p(\cdot)}(E)$ (see Definitions 3.1 and 3.2 for the definition of these sets). Then $u|_\Omega \in S_{p(\cdot),\Omega}(E)$ and

$$\int_\Omega \left|u|_\Omega\right|^{p(x)} dx + \int_\Omega \left|\nabla u|_\Omega\right|^{p(x)} dx \le \int_{\mathbb{R}^N} |u|^{p(x)} dx + \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx.$$

But the above inequality yields $\mathrm{Cap}_{p(\cdot),\Omega}(E) \le \mathrm{Cap}_{p(\cdot)}(E)$, as desired.

To establish the remaining inequality, first let $K \subseteq \overline{\Omega}$ be a compact set. Then by Proposition 3.7 there exists a a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \widetilde{W}^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega})$ such that

$$u_n \ge 1 \text{ on } K \qquad \text{and} \qquad \lim_{n\to\infty}\left(\int_\Omega |u_n|^{p(x)} dx + \int_\Omega |\nabla u_n|^{p(x)} dx\right) = \mathrm{Cap}_{p(\cdot),\Omega}(K).$$

Now let ε_p denote the extension operator used in the proof of Lemma 3.8, and put $v_n := \varepsilon_p(u_n)$. Then $v_n \in \widetilde{W}^{1,p(\cdot)}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $v_n \ge 1$ on K. It follows from Proposition 3.7, the continuity of ε_p, and [21, Lemma 3.2.5], that

$$\begin{aligned}
\mathrm{Cap}_{p(\cdot)}(K) &\le \int_{\mathbb{R}^N} |v_n|^{p(x)} dx + \int_{\mathbb{R}^N} |\nabla v_n|^{p(x)} dx \\
&\le \|v_n\|^{\hat{p}}_{W^{1,p(\cdot)}(\mathbb{R}^N)} \\
&\le C_{\varepsilon_p} \|u_n\|^{\hat{p}}_{W^{1,p(\cdot)}(\Omega)} \\
&\le C_{\varepsilon_p}\left(\int_\Omega |u_n|^{p(x)} dx + \int_\Omega |\nabla u_n|^{p(x)} dx\right)^{\hat{p}/\bar{p}} \xrightarrow{n\to\infty} C_{\varepsilon_p} \mathrm{Cap}_{p(\cdot),\Omega}(K)^{\hat{p}/\bar{p}},
\end{aligned}$$

for some constant $C_{\varepsilon_p} > 0$, where

$$\hat{p} := \begin{cases} p_*, & \text{if } \|v_n\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} \le 1, \\ p^*, & \text{if } \|v_n\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} > 1. \end{cases}$$

and

$$\bar{p} := \begin{cases} p_*, & \text{if } \|u_n\|_{W^{1,p(\cdot)}(\Omega)} > 1, \\ p^*, & \text{if } \|u_n\|_{W^{1,p(\cdot)}(\Omega)} \le 1. \end{cases}$$

This shows the first inequality in (3.2) for all $K \subseteq \overline{\Omega}$ compact. On the other hand, if $U \subseteq \overline{\Omega}$ is open, then there exists a increasing sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets such that $\bigcup_{n\in\mathbb{N}} K_n = U$. Then it follows from Proposition 3.4 that

$$\mathrm{Cap}_{p(\cdot)}(U) = \lim_{n\to\infty} \mathrm{Cap}_{p(\cdot)}(K_n) \le C_{\varepsilon_p} \lim_{n\to\infty} \mathrm{Cap}_{p(\cdot),\Omega}(K_n)^{\hat{p}/\bar{p}} = C_{\varepsilon_p}\mathrm{Cap}_{p(\cdot),\Omega}(U)^{\hat{p}/\bar{p}}.$$

Finally, if $E \subseteq \overline{\Omega}$ is an arbitrary set then by Proposition 3.4 we deduce that

$$\begin{aligned}
\mathrm{Cap}_{p(\cdot)}(E) &= \inf\left\{\mathrm{Cap}_{p(\cdot)}(U) \mid U \subseteq \mathbb{R}^N \text{ open, and } E \subseteq U\right\}\\
&= \inf\left\{\mathrm{Cap}_{p(\cdot)}(U \cap \overline{\Omega}) \mid U \subseteq \mathbb{R}^N \text{ open, and } E \subseteq U\right\}\\
&= \inf\left\{\mathrm{Cap}_{p(\cdot)}(W) \mid W \subseteq \overline{\Omega} \text{ open, and } E \subseteq W\right\}\\
&\le C_{\varepsilon_p} \inf\left\{\mathrm{Cap}_{p(\cdot),\Omega}(W)^{\hat{p}/\bar{p}} \mid W \subseteq \overline{\Omega} \text{ open, and } E \subseteq W\right\} = \mathrm{Cap}_{p(\cdot),\Omega}(E)^{\hat{p}/\bar{p}}.
\end{aligned}$$

This completes the proof of the first inequality in (3.2) in general, and thus the complete statement in (3.2) follows by letting $c_\Omega := C_{\varepsilon_p}$. □

The following result is an immediate consequence of the above theorem.

Corollary 3.10. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \le p^* < \infty$, and let $E \subseteq \overline{\Omega}$. Then E is $\mathrm{Cap}_{p(\cdot),\Omega}$-polar if and only if E is $\mathrm{Cap}_{p(\cdot),\mathbb{R}^N}$-polar.*

Next we investigate the relation between the relative $p(\cdot)$-capacity with the measures defined on the boundary of an open bounded set. We begin with the following definition.

Definition 3.11. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We say that a Borel measure μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-*admissible*, if $\mathrm{Cap}_{p(\cdot),\Omega}(\Gamma) = 0$ implies that $\mu(\Gamma) = 0$ for every Borel set $\Gamma \subseteq \partial\Omega$.

Given $p, r \in \mathcal{P}^{\log}(\overline{\Omega})$ with $1 \le p_* \le p^* < \infty$ and $1 \le r_* \le r^* < \infty$, and given μ a finite Borel regular measure supported on $\partial\Omega$, we deduce from the definition of the space $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu)$ that the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) \hookrightarrow L^{p(\cdot)}(\Omega, dx)$ is bounded. Also, if $p \in [1, N)$ and $r \in [1, p(N-1)(N-p)^{-1}]$ are constants and if $\mu = \mathcal{H}^{N-1}$, then it was obtained by Maz'ya [31] that the embedding $W_\mu(\Omega, \partial\Omega) \hookrightarrow L^{\frac{rN}{N-1}}(\Omega, dx)$ is continuous. This result has been generalized to variable exponent in [36] for $p, r \in C^{0,1}(\overline{\Omega})$ with $1 \le p_* \le p^* < N$ and $1 \le r(x) \le (N-1)p(x)(N-p(x))^{-1}$ for all $x \in \overline{\Omega}$. However, the above embeddings are not necessary injections. In fact, an example of a bounded domain such that the continuous embedding $W_\mu(\Omega, \partial\Omega) \hookrightarrow L^{p(\cdot)}(\Omega, dx)$ is not injective has been given in [4]. Hence we conclude this section by giving a necessary and sufficient condition for the above embeddings of variable exponent function spaces to be injections. The proof will run similar as the derivation for the constant case obtained in [10, Theorem 2.11], but we give full details of the proof for the sake of completeness.

Theorem 3.12. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain, let μ be a finite Borel regular measure on $\partial\Omega$, and let $p, r \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \le p^* < \infty$ and $1 \le r_* \le r^* < \infty$. Then the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) \hookrightarrow L^{p(\cdot)}(\Omega, dx)$ is an injection if and only if the measure μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible.*

Proof: First suppose that the measure μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible, and let $S : W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu) \to L^{p(\cdot)}(\Omega,dx)$ be the embedding operator, defined by $Sv := v|_\Omega$ for each $v \in W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu)$. We must show that S is injective. Indeed, let $u \in W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu)$ and assume that $Su = 0$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega})$ such that $u_n \overset{n\to\infty}{\longrightarrow} u$ in $W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu)$. Clearly $Su_n \overset{n\to\infty}{\longrightarrow} Su = 0$, and thus (by passing to a subsequence if necessary) we get that u_n converges to zero $p(\cdot)$-q.e. on $\overline{\Omega}$. Because μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible, it follows that u_n converges to zero μ-a.e. on $\partial\Omega$. But since $u_n|_{\partial\Omega} \overset{n\to\infty}{\longrightarrow} u|_{\partial\Omega}$ in $L^{r(\cdot)}(\partial\Omega,d\mu)$, we get $u = 0$ μ-a.e. on $\partial\Omega$ by the uniqueness of the limit, and therefore $u = 0$ as asserted.

To show the converse, assume that the measure μ is not $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible. Then there exists a Borel set $\Gamma \subseteq \partial\Omega$ such that $\mathrm{Cap}_{p(\cdot),\Omega}(\Gamma) = 0$ and $\mu(\Gamma) > 0$. Taking into account the inner regularity of μ, we may assume that Γ is compact. Because Γ is $\mathrm{Cap}_{p(\cdot),\Omega}$-polar, we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega})$ satisfying $0 \le u_n \le 1$, $u_n = 1$ on Γ, and $\lim\limits_{n\to\infty} \|u_n\|_{W^{1,p(\cdot)}(\Omega)} = 0$. For each $k \in \mathbb{N}$, put $U_k := \left\{x \in \mathbb{R}^N \mid \mathrm{dist}(x,\Gamma) < 1/k\right\}$. Then we see that

$$\Gamma \subseteq U_{k+1} \subseteq U_k \text{ for each } k \in \mathbb{N}, \quad \bigcap_{k\in\mathbb{N}} U_k = \Gamma, \quad \text{and } \lim_{k\to\infty} \mu(U_k \cap \partial\Omega) = \mu(\Gamma).$$

Next choose $v_k \in C_c^\infty(U_k)$ with $0 \le v_k \le 1$ and $v_k = 1$ over Γ. Then for each $n,k \in \mathbb{N}$ one sees that $u_n v_k \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega})$, $0 \le u_n v_k \le 1$, $u_n v_k = 1$ on Γ, and $\lim\limits_{n\to\infty} \|u_n v_k\|_{W^{1,p(\cdot)}(\Omega)} = 0$. Now set $w_k := u_{n_k} v_k$, where $n_k \in \mathbb{N}$ is chosen such that $\|w_k\|_{W^{1,p(\cdot)}(\Omega)} = \|u_{n_k} v_k\|_{W^{1,p(\cdot)}(\Omega)} \le 2^{-k}$. Then we have that $0 \le w_k \le 1$, $w_k = 1$ over Γ, $\lim\limits_{k\to\infty} w_k = 0$ in $W^{1,p(\cdot)}(\Omega)$, and $\lim\limits_{k\to\infty} w_k = \chi_\Gamma$ everywhere on $\overline{\Omega}$, where χ_E denotes the characteristic function over $E \subseteq \mathbb{R}^N$. But since $w_k = 1$ on Γ, it follows from [21, Lemma 3.2.12] that

$$\|w_k\|_{r(\cdot),\partial\Omega} \ge \frac{\mu(\Gamma)^{1/\hat{r}}}{2} > 0,$$

where

$$\hat{r} := \begin{cases} r_*, & \text{if } \|w_k\|_{r(\cdot),\partial\Omega} > 1, \\ r^*, & \text{if } \|w_k\|_{r(\cdot),\partial\Omega} \le 1. \end{cases}$$

Combining all this yields that $\chi_\Gamma \in W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu) \setminus \{0\}$, but $S(\chi_\Gamma) = 0$. Therefore the embedding $S : W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu) \to L^{p(\cdot)}(\Omega,dx)$ is not injective, as desired. □

4 Well-posedness of Robin and Wentzell-Robin Cauchy problems

In this section we state and prove the main results of this section. We will assume throughout this section that $\Omega \subseteq \mathbb{R}^N$ is a domain with finite measure, that μ is a finite Borel regular measure supported on $\partial\Omega$, and that $p \in \mathcal{P}^{\log}(\overline{\Omega})$ with $1 < p_* \le p^* < \infty$. We begin by defining the functionals associated with the $p(\cdot)$-Laplace equation with Robin boundary conditions, and also with Wentzell-Robin boundary conditions.

In fact, given $\beta \in L^\infty(\partial\Omega,d\mu)$ with $\inf\limits_{x\in\partial\Omega} \beta(x) \ge \beta_0$ for some constant $\beta_0 > 0$, we define the functional $\Phi_R : L^2(\Omega,dx) \to [0,\infty]$ by

$$\Phi_R(u) := \begin{cases} \frac{1}{p_*}\displaystyle\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{p_*}\int_{\partial\Omega} \beta|u|^{p(x)}\,d\mu, & \text{if } u \in D(\Phi_R), \\ +\infty, & \text{if } u \in L^2(\Omega,dx) \setminus D(\Phi_R), \end{cases} \tag{4.1}$$

where $D(\Phi_R) := W^1_{p(\cdot),p(\cdot)}(\Omega,\partial\Omega,d\mu)\cap L^2(\Omega,dx)$. This functional is associated with the Robin boundary value problem with variable exponent (see Proposition 4.3).

On the other hand, to define the functional associated with the Wentzell-Robin parabolic equation, we need to introduce some additional notations and definitions. Indeed, for functions $q, r \in \mathcal{P}(\overline{\Omega})$ with $q_* \geq 1$, $r_* \geq 1$, consider the vector space $\mathbb{X}^{q(\cdot),r(\cdot)}(\Omega,\partial\Omega) := L^{q(\cdot)}(\Omega,dx)\times L^{r(\cdot)}(\partial\Omega,d\mu)$, endowed with the norm

$$|||(u,v)|||_{q(\cdot),r(\cdot)} := \|(u,v)\|_{\mathbb{X}^{q(\cdot),r(\cdot)}(\Omega,\partial\Omega)} := \inf\left\{\lambda > 0 \mid \Lambda_{q,r}(u/\lambda, v/\lambda) \leq 1\right\},$$

if $q^* < \infty$ and $r^* < \infty$, where

$$\Lambda_{q,r}(u,v) := \int_\Omega |u|^{q(x)}\,dx + \int_{\partial\Omega} |v|^{r(x)}\,d\mu.$$

If $q(x) = r(x)$ over $\overline{\Omega}$, then we denote the above space, norm, and functional by $\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega)$, $|||(u,v)|||_{q(\cdot)}$, and Λ_q, respectively. Moreover, since $p^* < \infty$, we see that $\mathbb{X}^{p(\cdot)}(\Omega,\partial\Omega)$ is a Banach space and can be identified with $L^{p(\cdot)}(X,d\eta)$ for a suitable measure space (X,Σ,η) such that $L^\infty(X,d\eta)$ can be identified with $L^\infty(\Omega,dx)\times L^\infty(\partial\Omega,d\mu)$ with the norm

$$|||(u,v)|||_\infty := \max\left\{\|u\|_{\infty,\Omega}, \|v\|_{\infty,\partial\Omega}\right\},$$

for each $(u,v) \in L^\infty(\Omega,dx)\times L^\infty(\partial\Omega,d\mu)$.

Having said all this, given $\beta \in L^\infty(\partial\Omega,d\mu)$ with $\inf\limits_{x\in\partial\Omega}\beta(x) \geq \beta_0$ for some constant $\beta_0 > 0$, we define the functional $\Phi_W : \mathbb{X}^2(\Omega,\partial\Omega) \to [0,\infty]$ by

$$\Phi_W(u,v) := \begin{cases} \frac{1}{p_*}\displaystyle\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{p_*}\int_{\partial\Omega} \beta|u|^{p(x)}\,d\mu, & \text{if } (u,v) = (u,u|_{\partial\Omega}) \in D(\Phi_W), \\ +\infty, \quad \text{if } (u,v) \in \mathbb{X}^2(\Omega,\partial\Omega)\setminus D(\Phi_W), \end{cases} \tag{4.2}$$

with effective domain

$$D(\Phi_W) := \left\{(u,u|_{\partial\Omega}) \mid u \in W^1_{p(\cdot),p(\cdot)}(\Omega,\partial\Omega,d\mu)\cap L^2(\Omega,dx),\ u|_{\partial\Omega} \in L^2(\partial\Omega,d\mu)\right\}.$$

By definition it is clear that the functionals Φ_R and Φ_W are both proper and convex in $L^2(\Omega,dx)$ and $\mathbb{X}^2(\Omega,\partial\Omega)$, respectively.

The following result is the key step in the establishment of the main results. The proof follows a similar approach as in [41, Proof of Theorem 5.2].

Theorem 4.1. *The functionals Φ_R and Φ_W are both lower semicontinuous in their respective domains $L^2(\Omega,dx)$ and $\mathbb{X}^2(\Omega,\partial\Omega)$ if and only if the measure μ is* $\mathrm{Cap}_{p(\cdot),\Omega}$*-admissible.*

Proof: We prove only the semicontinuity of the functional Φ_W over $\mathbb{X}^2(\Omega,\partial\Omega)$; the semicontinuity of Φ_R in $L^2(\Omega,dx)$ can be obtained in a similar (and even simpler) manner.

We first suppose that μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible, and we take a sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}} := \{(u_n, u_n|_{\partial\Omega})\}_{n\in\mathbb{N}} \subseteq D(\Phi_W)$ such that $\lim\limits_{n\to\infty}\mathbf{u}_n = (u,w)$ in $\mathbb{X}^2(\Omega,\partial\Omega)$, that is, $u_n \xrightarrow{n\to\infty} u$ in $L^2(\Omega,dx)$ and $u_n|_{\partial\Omega} \xrightarrow{n\to\infty} w$ in $L^2(\partial\Omega,d\mu)$. If $\liminf\limits_{n\to\infty}\Phi_W(\mathbf{u}_n) = +\infty$ the conclusion is obvious, so we assume that $\liminf\limits_{n\to\infty}\Phi_W(\mathbf{u}_n) < +\infty$. Take a subsequence of $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$, which we also denote by $\mathbf{u}_n$, such that $\lim\limits_{n\to\infty}\Phi_W(\mathbf{u}_n)$ equals a constant. Now put

$$\mathbb{X}_\mu(\Omega,\partial\Omega) := \left(W^1_{p(\cdot),p(\cdot)}(\Omega,\partial\Omega,d\mu)\cap L^2(\Omega,dx)\right)\times L^2(\partial\Omega,d\mu),$$

and observe that $\mathbb{X}_\mu(\Omega,\partial\Omega)$ is a reflexive Banach space endowed with the norm

$$|||(u,v)|||_{\mathbb{X}_\mu(\Omega,\partial\Omega)} := \|u\|_{W^1_{p(\cdot),p(\cdot)}(\Omega,\partial\Omega,d\mu)} + \|u\|_{2,\Omega} + \|v\|_{2,\partial\Omega}.$$

Then we see that $\mathbf{u}_n$ is a bounded sequence in $\mathbb{X}_\mu(\Omega,\partial\Omega)$. Taking into account the convexity of $D(\Phi_w)$, we let $\mathbf{v}_n := (v_n, v_n|_{\partial\Omega})$ be a convex combination of $\mathbf{u}_n$, and such that $\lim_{n\to\infty} v_n = v$ in $W^1_{p(\cdot),p(\cdot)}(\Omega,\partial\Omega,d\mu)\cap L^2(\Omega,dx)$. Then $\lim_{n\to\infty}\mathbf{v}_n = \mathbf{v} := (v,\tilde{v})$ in $\mathbb{X}_\mu(\Omega,\partial\Omega)$, and $v_n \xrightarrow{n\to\infty} v$ in $\widetilde{W}^{1,p(\cdot)}(\Omega)$. By the uniqueness of the limit we see that $u = v$ a.e. on Ω, and moreover it follows (by taking a subsequence if necessary) that $v_n \xrightarrow{n\to\infty} v$ $p(\cdot)$-q.e. on $\partial\Omega$. But because μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible, the previous convergence implies that $v_n \xrightarrow{n\to\infty} v$ μ-a.e. on $\partial\Omega$. Since in addition $\lim_{n\to\infty} v_n = v$ in $L^{p(\cdot)}(\partial\Omega,d\mu)$, by virtue of the uniqueness of the limit we conclude that $\tilde{v} = v|_{\partial\Omega}$ and $v|_{\partial\Omega} = w$ μ-a.e. on $\partial\Omega$. But we also need to show that $w = u|_{\partial\Omega}$. Indeed, if $w \neq u|_{\partial\Omega}$, then since $\mathbf{v}_n \in D(\Phi_w)$, we see that $\Phi_w(\mathbf{v}_n) < +\infty$, but $\Phi_w(u,w) = +\infty$ by the definition of the functional Φ_w. However, by virtue of the Dominated Convergence Theorem one deduce that $+\infty = \Phi_w(u,w) = \lim_{n\to\infty}\Phi_w(\mathbf{v}_n) < +\infty$, which is clearly a contradiction. Hence $u|_{\partial\Omega} = v|_{\partial\Omega} = w$ μ-a.e. on $\partial\Omega$, and thus $\mathbf{v} = (u,w) = (u,u|_{\partial\Omega}) := \mathbf{u}$ η-a.e. on $\Omega\times\partial\Omega$. Finally, the convexity of Φ_w entails that

$$\Phi_w(\mathbf{u}) = \liminf_{n\to\infty}\Phi_w(\mathbf{v}_n) \leq \liminf_{n\to\infty}\Phi_w(\mathbf{u}_n),$$

and thus Φ_w is semicontinuous over $\mathbb{X}^2(\Omega,\partial\Omega)$.

Assume now that μ is not $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible. Then by virtue of the inner regularity of μ together with the assumption we may suppose that there exists a compact set $\Gamma\subseteq\partial\Omega$ such that $\mathrm{Cap}_{p(\cdot),\Omega}(\Gamma) = 0$, but $\mu(\Gamma) > 0$. Now let $\{w_k\}_{k\in\mathbb{N}}$ denote the sequence defined in the proof of Theorem 3.12. Then we recall that $0\leq w_k\leq 1$, $w_k = 1$ over Γ, $\lim_{k\to\infty} w_k = 0$ in $W^{1,p(\cdot)}(\Omega)\cap C_c(\overline{\Omega})$, and $\lim_{k\to\infty} w_k = \chi_\Gamma$ everywhere on $\overline{\Omega}$. Moreover, we see that $\lim_{k\to\infty} w_k = 0$ in $L^2(\Omega,dx)$. Without loss of generality, we may suppose that $\{w_k\}_{k\in\mathbb{N}}$ is a decreasing function. Now for each $k\in\mathbb{N}$, we put $\hat{w}_k := w_1 - w_k$ and $\hat{\mathbf{w}}_k := (\hat{w}_k, \hat{w}_k|_{\partial\Omega})$. Then we see that $\hat{w}_k \xrightarrow{k\to\infty} w_1\chi_{\partial\Omega\setminus\Gamma}$ everywhere, and $0\leq\hat{w}_k\leq w_1$, and thus we deduce that

$$\int_{\partial\Omega\setminus\Gamma}\beta|w_1|^{p(x)}\,d\mu = \liminf_{k\to\infty}\int_{\partial\Omega}\beta|\hat{w}_k|^{p(x)}\,d\mu.$$

If $w_1\chi_{\partial\Omega\setminus\Gamma} = w_1|_{\partial\Omega}$, then using the fact that $w_1 = 1$ over Γ together with the convergence $\hat{\mathbf{w}}_k \xrightarrow{k\to\infty} (w_1, w_1\chi_{\partial\Omega\setminus\Gamma})$ in $\mathbb{X}^2(\Omega,\partial\Omega)$ and $\nabla\hat{w}_k \xrightarrow{k\to\infty}\nabla w_1$ in $L^{p(\cdot)}(\Omega,dx)$, we calculate to get that

$$\begin{aligned}
\Phi_w(\mathbf{w}_1) &= \frac{1}{p_*}\int_\Omega|\nabla w_1|^{p(x)}\,dx + \frac{1}{p_*}\int_{\partial\Omega}\beta|w_1|^{p(x)}\,d\mu\\
&\geq \frac{1}{p_*}\int_\Omega|\nabla w_1|^{p(x)}\,dx + \frac{1}{p_*}\int_{\partial\Omega\setminus\Gamma}\beta|w_1|^{p(x)}\,d\mu + \frac{\beta_0}{p_*}\mu(\Gamma)\\
&> \frac{1}{p_*}\int_\Omega|\nabla w_1|^{p(x)}\,dx + \frac{1}{p_*}\int_{\partial\Omega\setminus\Gamma}\beta|w_1|^{p(x)}\,d\mu = \liminf_{k\to\infty}\Phi_w(\hat{\mathbf{w}}_k).
\end{aligned}$$

On the other hand, if $w_1\chi_{\partial\Omega\setminus\Gamma}\neq w_1|_{\partial\Omega}$, then by definition $\Phi_w(w_1, w_1\chi_{\partial\Omega\setminus\Gamma}) = +\infty$, but $\hat{\mathbf{w}}_k\in D(\Phi_w)$ and $\Phi_w(\hat{\mathbf{w}}_k) < +\infty$ for each $k\in\mathbb{N}$. Therefore in both cases we have shown that Φ_w is not lower semicontinuous, as required. □

Next, before proceeding with our discussion, we make the following observation. Indeed, because the framework of our interest include non-smooth and "bad" domains, we will need to introduce a notion of normal derivative in the weak sense for Sobolev functions, which has been discussed in [10] for the constant case, and defined in [36] for variable exponents.

Definition 4.2. Let $p \in \mathcal{P}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < \infty$, let η be a Borel measure supported on $\partial\Omega$, and let $u \in W^{1,1}_{loc}(\Omega)$ be such that $|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \in L^1(\Omega, dx)$ for all $v \in C^1(\overline{\Omega})$. If there exists a function $f \in L^1_{loc}(\mathbb{R}^N, dx)$ such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx = \int_\Omega f v\, dx + \int_{\partial\Omega} v\, d\eta,$$

for all $v \in C^1(\overline{\Omega})$, then we say that η is the $p(\cdot)$-*generalized normal derivative* of u, and we denote

$$|\nabla u|^{p(\cdot)-2}\frac{\partial u}{\partial \nu} d\mathcal{H}^{N-1} := \eta.$$

As mentioned before, recall that if Ω is "sufficiently regular", for instance, a bounded Lipschitz domain, then it is well-known that Ω is a $W^{1,p(\cdot)}$-extension domain, and that the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$ is an upper and lower $(N-1)$-Ahlfors measure on $\partial\Omega$, which coincides with the classical surface measure on $\partial\Omega$. Hence in this case the notion of generalized normal derivative coincide with the classical definition of the normal derivative.

Now we compute the subdifferential of the functionals $\Phi_{_R}$ and $\Phi_{_W}$.

Proposition 4.3. *Let $\Omega \subseteq \mathbb{R}^N$ is a domain with finite measure, let μ be a finite Borel regular* $\mathrm{Cap}_{p(\cdot),\Omega}$-*admissible measure supported on $\partial\Omega$, and let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < \infty$.*

(a) *Let $\partial\Phi_{_R}$ be the subdifferential associated with $\Phi_{_R}$, and let $f \in L^2(\Omega, dx)$ and $u \in D(\Phi_{_R})$. Then $f \in \partial\Phi_{_R}(u)$ if and only if*

$$\begin{cases} -\Delta_{p(\cdot)} u = f \qquad \text{in } \mathcal{D}(\Omega)^*, \\ |\nabla u|^{p(\cdot)-2}\dfrac{\partial u}{\partial \nu} d\mathcal{H}^{N-1} + \beta |u|^{p(\cdot)-2} u\, d\mu = 0 \;\; \text{weakly on } \partial\Omega, \end{cases} \tag{4.3}$$

where $\Delta_{p(\cdot)} u := \mathrm{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$ *denotes the generalized $p(\cdot)$-Laplace operator on Ω.*

(b) *Let $\partial\Phi_{_W}$ be the subdifferential associated with $\Phi_{_W}$, and let $\mathbf{f} := (f, f|_{\partial\Omega}) \in \mathbb{X}^2(\Omega, \partial\Omega)$ and $\mathbf{u} := (u, u|_{\partial\Omega}) \in D(\Phi_p)$. Then $\mathbf{f} \in \partial\Phi_p(\mathbf{u})$ if and only if*

$$\begin{cases} \Delta_{p(\cdot)} u = f \qquad \text{in } \mathcal{D}(\Omega)^*, \\ -\Delta_{p(\cdot)} u\, d\mu + |\nabla u|^{p(\cdot)-2}\dfrac{\partial u}{\partial \nu} d\mathcal{H}^{N-1} + \beta |u|^{p(\cdot)-2} u\, d\mu = 0 \;\; \text{weakly on } \partial\Omega. \end{cases} \tag{4.4}$$

where at the boundary, $\Delta_{p(\cdot)} u := (\Delta p(\cdot) u)|_{\partial\Omega}$ stands as the restriction of the $p(\cdot)$-Laplace operator to the boundary $\partial\Omega$.

Proof: We only give the proof of part (b), for part (a) follows in a similar and even easier way. For simplicity, for each function w with a well-defined trace $w|_{\partial\Omega}$ at the boundary, we will write

$\mathbf{w} := (w, w|_{\partial\Omega})$.

To establish part (b), first suppose that $\mathbf{f} \in \partial\Phi_W(\mathbf{u})$, for $\mathbf{u} \in D(\Phi_W)$. Then for all $\mathbf{v} \in D(\Phi_W)$ one gets

$$\begin{aligned}\int_\Omega f(v-u)\,dx + \int_{\partial\Omega} f(v-u)\,d\mu \\ \leq \frac{1}{p_*}\int_\Omega \left(|\nabla v|^{p(x)} - |\nabla u|^{p(x)}\right)dx + \frac{1}{p_*}\int_{\partial\Omega} \beta\left(|v|^{p(x)} - |u|^{p(x)}\right)d\mu.\end{aligned} \tag{4.5}$$

Fix $\mathbf{w} \in D(\Phi_W)$ and $t \in (0,1]$. Then substituting $\mathbf{v}$ by $t\mathbf{w} + (1-t)\mathbf{u} \in D(\Phi_W)$ in (4.5), dividing by t and taking the limit as $t \downarrow 0^+$, we obtain that

$$\begin{aligned}\int_\Omega f(v-u)\,dx + \int_{\partial\Omega} f(v-u)\,d\mu \\ \leq \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(v-u)\,dx + \int_{\partial\Omega} \beta|u|^{p(x)-2}u(v-u)\,d\mu.\end{aligned} \tag{4.6}$$

Moreover, for every $\Psi := (\psi, \psi|_{\partial\Omega}) \in \widetilde{D(\Phi_W)} := \{\mathbf{u} \in D(\Phi_W) \mid u \in C(\overline{\Omega})\}$, replacing $\mathbf{w}$ by $\mathbf{u} \pm \Psi$ in (4.6) gives

$$\int_\Omega f\psi\,dx + \int_{\partial\Omega} f\psi\,d\mu = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\psi\,dx + \int_{\partial\Omega} \beta|u|^{p(x)-2}u\psi\,d\mu, \tag{4.7}$$

for all $\Psi \in \widetilde{D(\Phi_W)}$. Letting $\psi \in \mathcal{D}(\Omega)$ in (4.7) we deduce that $f = \Delta_{p(\cdot)}u$; letting then $\psi \in C^1(\overline{\Omega})$ and applying partial integration in (4.7) yields that $f|_{\partial\Omega}\,d\mu = -|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}d\mathcal{H}^{N-1} - \beta|u|^{p(x)-2}u\,d\mu$ weakly over $\partial\Omega$ (in the sense of Definition 4.2). This facts give the assertion (4.4).

To show the converse, let $\mathbf{u} \in D(\Phi_W)$, $\mathbf{f} \in \mathbb{X}^2(\Omega, \partial\Omega)$, and suppose that u fulfills the equation (4.4). Using the well-known property $\frac{1}{r}(|a|^r - |b|^r) \geq |b|^{r-2}b(a-b)$, which is valid for all $a, b \in \mathbb{R}^N$, and for every $r \in (1, \infty)$, we deduce that

$$\Phi_W(\mathbf{v}) - \Phi_W(\mathbf{u}) \geq \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(v-u)\,dx + \int_{\partial\Omega} \beta|u|^{p(x)-2}u(v-u)\,d\mu, \tag{4.8}$$

for every $\mathbf{v} \in D(\Phi_W)$. From (4.8) and the definition of weak solution of (4.4) it follows that

$$\Phi_W(\mathbf{v}) - \Phi_W(\mathbf{u}) \geq \int_\Omega f(v-u)\,dx + \int_{\partial\Omega} f(v-u)\,d\mu, \tag{4.9}$$

for every $\mathbf{v} \in \widetilde{D(\Phi_W)}$, and thus for all $\mathbf{v} \in D(\Phi_W)$. Therefore $\mathbf{f} \in \partial\Phi_W(\mathbf{u})$, completing the proof of part (b). □

Now we state the result about the well-posedness of the parabolic equations of variable exponent type with either Robin or Wentzell-Robin boundary conditions on arbitrary domains.

Theorem 4.4. *Assume all the conditions of Proposition 4.3.*

(a) *The operator $\mathcal{A}_R := \partial\Phi_R$ generates an order-preserving submarkovian C_0-semigroup $\{T_R(t)\}_{t\geq 0}$ on $L^2(\Omega, dx)$. Moreover, the semigroup $\{T_R(t)\}_{t\geq 0}$ is non-expansive over $L^{q(\cdot)}(\Omega, dx)$ for each $q \in \mathcal{P}(\overline{\Omega})$ such that $1 \leq q_* \leq q^* \leq \infty$. Consequently, for each $q \in \mathcal{P}(\overline{\Omega})$ with $1 \leq q_* \leq q^* < \infty$, and for each $u_0 \in L^{q(\cdot)}(\Omega, dx)$, the function $u(\cdot) = T_R(\cdot)u_0$ is the (unique) strong solution of the Cauchy problem with Robin boundary conditions*

$$\begin{cases} u \in C(\mathbb{R}_+; L^{q(\cdot)}(\Omega, dx)) \cap W^{1,\infty}_{loc}((0,\infty); L^{q(\cdot)}(\Omega, dx)) \text{ and } u(t) \in \partial\Phi_R \text{ a.e.}, \\ \dfrac{\partial u}{\partial t} + \partial\Phi_R(u) = 0 \text{ a.e. on } \mathbb{R}_+, \\ u(0,x) = u_0(x). \end{cases} \tag{4.10}$$

(b) *The operator $\mathcal{A}_{W} := \partial\Phi_{W}$ generates an order-preserving submarkovian C_0-semigroup $\{T_{W}(t)\}_{t\geq 0}$ on $\mathbb{X}^2(\Omega,\partial\Omega)$. Moreover, the semigroup $\{T_{W}(t)\}_{t\geq 0}$ is non-expansive over $\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega)$ for each $q\in\mathcal{P}(\overline{\Omega})$ such that $1\leq q_*\leq q^*\leq\infty$. Consequently, for each $q\in\mathcal{P}(\overline{\Omega})$ with $1\leq q_*\leq q^*<\infty$, and for each $\mathbf{u}_0 := (u_0, u_0|_{\partial\Omega})\in\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega)$, the function $\mathbf{u}(\cdot) = T_{W}(\cdot)\mathbf{u}_0$ is the (unique) strong solution of the Cauchy problem with Wentzell-Robin boundary conditions*

$$\begin{cases} \mathbf{u}\in C(\mathbb{R}_+;\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega))\cap W^{1,\infty}_{loc}((0,\infty);\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega)) \text{ and } \mathbf{u}(t)\in\partial\Phi_{W} \text{ a.e.,} \\ \dfrac{\partial\mathbf{u}}{\partial t}+\partial\Phi_{W}(\mathbf{u}) = 0 \text{ a.e. on } \mathbb{R}_+, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x). \end{cases} \tag{4.11}$$

Proof: Once again it only suffices to show part (b). In fact, since $p\in\mathcal{P}^{\log}(\overline{\Omega})$ with $1<p_*\leq p^*<\infty$, we see that $D(\Phi_{W})$ is dense in $\mathbb{X}^2(\Omega,\partial\Omega)$, and hence by Theorem 2.10, the operator $\mathcal{A}_{W} := \partial\Phi_{W}$ generates a (nonlinear) C_0-semigroup $\{T_{W}(t)\}_{t\geq 0}$ on $\mathbb{X}^2(\Omega,\partial\Omega)$. Thus, for each $\mathbf{u}_0\in\mathbb{X}^2(\Omega,\partial\Omega)$, the function $\mathbf{u}(\cdot) := T_{W}(\cdot)\mathbf{u}_0$ fulfills the Eq. (4.11). To complete the proof, it suffices to show that $\{T_{W}(t)\}_{t\geq 0}$ is order-preserving and non-expansive. Given $(u_1,v_1),(u_2,v_2)\in\mathbb{X}^2(\Omega,\partial\Omega)$, if (u_1,v_1) or (u_2,v_2) does not belong to $D(\Phi_{W})$, the conclusion is obvious. If $(u_1,v_1),(u_2,v_2)\in D(\Phi_{W})$, then to prove the order-preserving property, since $W^{1,p(\cdot)}(\Omega)$ is a lattice (e.g. [21], Proposition 8.1.9), letting

$$g_{u,v} := \frac{1}{2}(u+u\wedge v) \qquad \text{and} \qquad h_{u,v} := \frac{1}{2}(v+u\vee v)$$

for each $u,v\in W^{1,p(\cdot)}(\Omega)$, we have that both $\mathbf{g}_{u,v}$ and $\mathbf{h}_{u,v}$ lie in $D(\Phi_{W})$. Moreover, using the well-known property $|a+b|^r\leq 2^{r-1}(|a|^r+|b|^r)$, valid for all $a,b\in\mathbb{R}^N$ and for all $r\in[1,\infty)$, one has

$$\begin{aligned}\Phi_{W}(\mathbf{g}_{u,v})+\Phi_{W}(\mathbf{h}_{u,v}) \leq{}& \frac{1}{p_*}\int_{\Omega\cap\{u\leq v\}}(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx+\frac{1}{p_*}\int_{\partial\Omega\cap\{u\leq v\}}(|u|^{p(x)}+|v|^{p(x)})\,d\mu+ \\ &+\frac{2}{p_*}\int_{\Omega\cap\{u>v\}}\left(\frac{|\nabla(u+v)|}{2}\right)^{p(x)}dx+\frac{2}{p_*}\int_{\partial\Omega\cap\{u>v\}}\left(\frac{|u+v|}{2}\right)^{p(x)}d\mu \leq \Phi_{W}(\mathbf{u})+\Phi_{W}(\mathbf{v}).\end{aligned}$$

Thus $\left\{T_{p(\cdot)}(t)\right\}_{t\geq 0}$ is order-preserving by virtue of Proposition 2.12. Moreover, given $\alpha>0$, put

$$g_{u,v,\alpha} := \frac{1}{2}[(u-v+\alpha)^+-(u-v-\alpha)^-]\in W^{1,p(\cdot)}(\Omega)$$

and notice that

$$\begin{aligned}&\Phi_{W}(\mathbf{v}+\mathbf{g}_{u,v,\alpha})+\Phi_{W}(\mathbf{u}-\mathbf{g}_{u,v,\alpha}) \\ &\leq \frac{1}{p_*}\int_{\Omega\cap\{|u-v|\leq\alpha\}}(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx+\frac{1}{p_*}\int_{\partial\Omega\cap\{|u-v|\leq\alpha\}}(|u|^{p(x)}+|v|^{p(x)})\,d\mu+ \\ &\quad+\frac{2}{p_*}\int_{\Omega\cap\{|u-v|>\alpha\}}\left(\frac{|\nabla(u+v)|}{2}\right)^{p(x)}dx+\frac{2}{p_*}\int_{\partial\Omega\cap\{|u-v|>\alpha\}}\left(\frac{|u+v|}{2}\right)^{p(x)}d\mu \leq \Phi_{W}(\mathbf{u})+\Phi_{W}(\mathbf{v}).\end{aligned}$$

Hence it follows from Proposition 2.13 that $\{T_{W}(t)\}_{t\geq 0}$ is submarkovian. By [11, Theorem 1] and [30, Corollary 3] we see that $\{T_{W}(t)\}_{t\geq 0}$ can be extended to a non-expansive semigroup on $\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega)$ for every $q\in\mathcal{P}(\overline{\Omega})$ with $1\leq q_*\leq q^*<\infty$. To see the strong continuity of $\{T_{W}(t)\}_{t\geq 0}$ over $\mathbb{X}^{q(\cdot)}(\Omega,\partial\Omega)$, first take $\mathbf{u}\in\mathbb{X}^2(\Omega,\partial\Omega)\cap\mathbb{X}^\infty(\Omega,\partial\Omega)$, let $\hat{q} := \min\{2,q_*\}$, and let $r(x) := \frac{\hat{q}}{q(x)}\left(\frac{q^*-q(x)}{q^*-\hat{q}}\right)$, $s(x) := \frac{q^*}{q(x)}\left(\frac{q(x)-\hat{q}}{q^*-\hat{q}}\right)$

for all $x \in \overline{\Omega}$. Then by [29, Corollary 2.2] and Hölder's inequality (if necessary) we obtain that for $t > 0$,

$$\left|\left|\left|T_W(t)\mathbf{u} - \mathbf{u}\right|\right|\right|_{q(\cdot)} \leq c \left|\left|\left|T_W(t)\mathbf{u} - \mathbf{u}\right|\right|\right|_2^{\gamma_1} \cdot \left|\left|\left|T_W(t)\mathbf{u} - \mathbf{u}\right|\right|\right|_\infty^{\gamma_2} \xrightarrow{t \to 0^+} 0, \tag{4.12}$$

where

$$\gamma_1 := \begin{cases} r^*, & \text{if } \left|\left|\left|T_W\mathbf{u} - \mathbf{u}\right|\right|\right|_2 > 1, \\ r_*, & \text{if } \left|\left|\left|T_W\mathbf{u} - \mathbf{u}\right|\right|\right|_2 \leq 1, \end{cases} \quad \text{and} \quad \gamma_2 := \begin{cases} s^*, & \text{if } \left|\left|\left|T_W\mathbf{u} - \mathbf{u}\right|\right|\right|_{q^*} > 1, \\ s_*, & \text{if } \left|\left|\left|T_W\mathbf{u} - \mathbf{u}\right|\right|\right|_{q^*} \leq 1. \end{cases}$$

Finally, for a general $\mathbf{u} \in \mathbb{X}^{q(\cdot)}(\Omega, \partial\Omega)$, fix $\epsilon > 0$ and choose $\mathbf{v} \in \mathbb{X}^2(\Omega, \partial\Omega) \cap \mathbb{X}^\infty(\Omega, \partial\Omega)$ such that $|||\mathbf{u} - \mathbf{v}|||_{q(\cdot)} < \epsilon/3$. Then for a sufficiently small $t > 0$, we get from (4.12) that

$$\left|\left|\left|T_W(t)\mathbf{u} - \mathbf{u}\right|\right|\right|_{q(\cdot)} \leq \left|\left|\left|T_W(t)\mathbf{u} - T_W(t)\mathbf{v}\right|\right|\right|_{q(\cdot)} + \left|\left|\left|T_W(t)\mathbf{v} - \mathbf{v}\right|\right|\right|_{q(\cdot)} + |||\mathbf{u} - \mathbf{v}|||_{q(\cdot)} < \epsilon. \tag{4.13}$$

Since $\epsilon > 0$ was chosen arbitrary, the inequality (4.13) implies that $\{T_W(t)\}_{t \geq 0}$ is a C_0-semigroup over $\mathbb{X}^{q(\cdot)}(\Omega, \partial\Omega)$, and this completes the proof. □

Next we combine all the results previously established to state (without proof) the following result, which summarizes the majority of the previous results into one.

Theorem 4.5. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain with finite measure, let μ be a finite Borel regular measure supported on $\partial\Omega$, and let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < \infty$. The following assertions are equivalent.*

(a) *The measure μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible.*

(b) *the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) \hookrightarrow L^{p(\cdot)}(\Omega, dx)$ is an injection.*

(c) *The parabolic problem with variable exponent and Robin boundary conditions*

$$\begin{cases} u_t - \Delta_{p(\cdot)}u = 0 \, in \ \Omega \times (0, \infty), \\ |\nabla u|^{p(\cdot)-2} \dfrac{\partial u}{\partial \nu} d\mathcal{H}^{N-1} + \beta |u|^{p(\cdot)-2} u \, d\mu = 0 \ \ on \ \partial\Omega \times (0, \infty), \\ u(0, x) = u_0(x) \ in \ \Omega \times \partial\Omega, \end{cases} \tag{4.14}$$

is well-posed in $L^{q(\cdot)}(\Omega, dx)$ for each $q \in \mathcal{P}(\overline{\Omega})$ such that $1 \leq q_* \leq q^* < \infty$.

(d) *The parabolic problem with variable exponent and Wentzell-Robin boundary conditions*

$$\begin{cases} u_t - \Delta_{p(\cdot)}u = 0 \quad in \ \Omega \times (0, \infty), \\ -\Delta_{p(\cdot)}u \, d\mu + |\nabla u|^{p(\cdot)-2} \dfrac{\partial u}{\partial \nu} d\mathcal{H}^{N-1} + \beta |u|^{p(\cdot)-2} u \, d\mu = 0 \ \ on \ \partial\Omega \times (0, \infty), \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) \quad in \ \Omega \times \partial\Omega, \end{cases} \tag{4.15}$$

is well-posed on $\mathbb{X}^{q(\cdot)}(\Omega, \partial\Omega)$ for each $q \in \mathcal{P}(\overline{\Omega})$ such that $1 \leq q_* \leq q^* < \infty$.

The next example illustrates some situations where the above parabolic problems are well-posed, even on non-smooth domains.

Example 4.6. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, and let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < \infty$.

(a) By [21, Proposition 10.4.2], for any $E \subseteq \overline{\Omega}$, if $\mathrm{Cap}_{p(\cdot)}(E) = 0$, then $\mathcal{H}^s(E) = 0$ for every $s > N - p_*$, where we recall that $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure. Thus, if $\partial\Omega$ is finite with respect to $\mathcal{H}^s$ for some $s > N - p_*$, then it follows from the above comment together with Corollary 3.10 that $\mathcal{H}^s$ is a $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible measure. Hence in this case all the conclusions in Theorem 4.5 are valid.

(b) If μ is an upper d-Ahlfors measure supported on $\partial\Omega$ for $d \in (N - p_*, N)$, then it follows from [7, Remark 6.5] that $\mu(\Gamma) = 0$ for every $\Gamma \subseteq \partial\Omega$ with $\mathcal{H}^d(\Gamma) = 0$. Henceforth this observation together with the discussion in part (a) clearly imply that μ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible. Moreover, in this case we can show that the continuous embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\Omega, dx)$ discussed in Remark 2.5 is also an injection. Note that in Example 2.6 we have given two concrete examples of domains and boundary measures where this conclusion is valid, and hence where the conclusions of Theorem 4.5 are all fulfilled.

(c) Suppose that μ is a lower d-Ahlfors measure supported on $\partial\Omega$ for some $d \in [0, N)$ that is also $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible. Then by [17, Theorem 7.11] we deduce that $\mathcal{H}^d$ is also $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible.

We conclude our discussion with the following observation, discussed in [41, Remark 4.2] for the constant case.

Remark 4.7. Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary domain, and consider the space $W_\mu(\Omega, \partial\Omega)$ discussed in Remark 2.5 for $\mu = \mathcal{H}^{N-1}$. Then it may happen that μ is not a finite measure at the boundary $\partial\Omega$. In fact, an example of this situation is when Ω denotes the snowflake curve discussed in Example 2.6. However, in this case we can obtain the conclusions of Theorem 4.5. To justify this, let

$$\Gamma_\infty := \left\{ x \in \partial\Omega \mid \mathcal{H}^{N-1}(B_r(x) \cap \partial\Omega) = \infty, \text{ for all } r > 0 \right\}.$$

Then $\Gamma_\infty \subseteq \partial\Omega$ is relative closed, and it follows (as in the constant case) that every function $u \in \left\{ w \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid \int_{\partial\Omega} |w|^{p(x)}\, d\mathcal{H}^{N-1} < \infty \right\}$ satisfies $u|_{\Gamma_\infty} = 0$. In addition, because the closure of the set $\left\{ u \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid u|_{\Gamma_\infty} = 0 \right\}$ is the space $\left\{ u \in \widetilde{W}^{1,p(\cdot)}(\Omega) \mid \tilde{u} = 0 \;\; p(\cdot)\text{-q.e. on } \Gamma_\infty \right\}$, it follows that functions in $W_\mu(\Omega, \partial\Omega)$ (for $\mu = \mathcal{H}^{N-1}$) vanish $p(\cdot)$-q.e. on Γ_∞. Now, letting $\Gamma_0 := \partial\Omega \setminus \Gamma_\infty$, it follows that $\Gamma_0 \subseteq \partial\Omega$ is relative open, and on it we have that $\mathcal{H}^{N-1}$ is locally finite. Then if in addition Ω is a bounded $W^{1,p(\cdot)}$-extension domain, then it follows from part (a) in Example 4.6 that $\mathcal{H}^{N-1}$ is $\mathrm{Cap}_{p(\cdot),\Omega}$-admissible, but $\mathcal{H}^{N-1}$ may not be finite over $\partial\Omega$. However, the discussion above shows that in this case all the conclusions of Theorem 4.5 can be achieved.

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