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Continuous-Time Markov Processes as a Stochastic Model for Sedimentation[1]

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Markov processes with a continuous-time parameter are more satisfactory for describing sedimentation than discrete-time Markov chains because they treat sedimentation as a natural process that happens continuously (i.e., which is unbroken in time). They also avoid certain technicalities that arise in discrete time—namely, the choice of a time unit. Finally, they yield not only the same information as a discrete-time analysis, but also give information about the distribution of the thicknesses of the lithologies.

KEY WORDS: Markov process, lithology, Markov renewal process.

INTRODUCTION

In the past, various researchers (e.g., Gingerich 1969; Ethier, 1975; Powers and Easterling, 1982; Carr, 1982), have used the theory of discrete-time Markov chains to describe the structure of a sequence of lithologies. A Markov process is a stochastic process, meaning a sequence of random events, in which the only information useful for predicting the state of the sequence at time n contained in the history of the process (i.e., the sequence of states visited before time n) is the last state observed:

$$P(Y_n = j | Y_{n-1} = i, \ldots, Y_0 = i_0) = P(Y_n = j | Y_{n-1} = i) = q_{ij}$$

Here n, a natural number, is a discrete-time parameter, Y_n is the random variable describing the state the process occupies at time n, i, i_0 to i_{n-1} and j are elements of the state space (e.g., the rock types of a stratigraphic column). The matrix Q with entries q_{ij} is called the transition matrix of the process. It gives the probability q_{ij} of a transition from state i to state j.

This Markov chain is called time-homogeneous because the transition matrix Q does not depend on the time parameter n. This means the stochastic structure is the same throughout the time of the evolution of the system. Ho-

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mogeneity within the observed column is an essential assumption for the theory described here, although in reality it may not always be fulfilled.

Processes with a "memory" longer than the one considered—in particular, processes where the knowledge of a greater but fixed part of the history of the process, here, the sequence of lithologies in the stratigraphic column, is helpful—are so-called multistage Markov processes. Although processes of such a type will not be discussed in this paper, the theory can easily be extended to that case. The statistical analysis of multistage Markov chains has been treated by Chatfield (1973).

One of the problems arising in the modeling of sedimentary sequences as a discrete-time Markov chain is the definition of one unit, either of time or of the thickness of a lithology. Ethier (1975) showed that the choice of a fixed unit leads to transition matrices with overly large frequences on the diagonal ($i = j$), which means that "changes" from a rock type to itself tend to be overrepresented.

Another possibility is to count the number of transitions from one rock type to another, disregarding the thickness of the layers. This approach leads to the so-called embedded Markov chains, which have zeroes on the diagonal. Those zeroes are structural because they do not reflect probabilities but are a result of the mathematical model. A statistical test for the hypothesis of independence in the stochastic process (i.e., the independence of the process from its history, given the structural zeroes) has been described by Goodman (1968), and was introduced in the geological literature by Powers and Easterling (1982).

Although the model described in this paper is essentially different from previous ones, this test will also be of use for the statistical analysis of continuous-time Markov processes.

A third method to attack this problem has not as yet been discussed in the literature, and that is to use continuous-time Markov processes as a model. These are processes with a continuous-time parameter which also have the Markovian property that, given the present state, the future is independent of the past. "Continuous" does not mean uniform, but unbroken, and it only assumes that sedimentation does not happen instantaneously. "Time" in our context will mean the thickness of a sedimentary segment. For this analysis to be valid, one has to assume that the sedimentation has reached a position where subsequent erosion is not likely.

The thickness of a segment of the stratigraphic column is a random variable. If the continuous-time Markov model fits, then the theory of Markov processes show that the sequence of thicknesses of every rock type is "memoryless" (i.e., it is a sequence of independent exponential random variables. For a mathematical derivation of this fact, see Karlin and Taylor (1975).

To establish that a stratigraphic column follows a continuous-time Markov process, two steps are necessary. In the first step, one shows that the sequence of rock types follows a discrete Markov chain. This step is exactly the procedure described by Powers and Easterling (1982), and it yields the same information. In the second step, each rock type is analyzed separately as to whether its thicknesses have an exponential distribution. The following artificial example shows one way in which this additional information can be used—namely, to reconstruct parts of the stratigraphic column.

The analysis of a column consisting of three rock types, named A, B, C, showed that it follows a Markov process with some transition matrix Q and exponentially distributed thicknesses with parameters 1 for A, 5 for B, and 10 for C. A small segment of the column is as follows:

.. , A 2.0 units, B 1.5 units, unreadable 0.5 units, B .8 units, ..

Then the exponential distribution of the thickness of B yields that the probability that the unreadable part was in fact all type B is at least 0.87.

Analogous to the Markov chain model, transition probabilities and expected thicknesses of the segments completely describe the continuous-time Markov process.

Use of this model not only frees one from the necessity of choosing one unit of time, but also seems to fit the problem much better, because it describes sedimentary sequences as being created continuously in time (i.e., unbroken), with time being an essential part of the process.

CONTINUOUS-TIME MARKOV PROCESSES

A brief introduction into the theory of continuous-time Markov processes is given here. For details, see Karlin and Taylor (1975) and Cinlar (1975).

A stochastic process $\{X_t, t \geq 0\}$ is called a continuous-time Markov process with discrete state space S, provided that for any t, $s \geq 0$, and $j \in S$

$$P\{X_{t+s} = j | X_u, u \leq t\} = P\{X_{t+s} = j | X_t\}$$

In the case described here, S is the set of rock types, t is the thickness of the column, and X_t is the rock type at thickness t in the column.

A macroscopic way to describe a continuous-time Markov process is to look at probabilities of changes from one rock type to another, and at the thickness of each segment. The sequence of changes forms a discrete-time Markov chain, called the underlying Markov chain, and is best described by a transition matrix Q. As already stated, a transition matrix Q completely describes the evolution of a discrete-time Markov chain. The process associated with the matrix Q is precisely the one studied by previous researchers (e.g., Powers and Easterling, 1982). On the other hand, the thickness of each segment of a rock type has an exponential distribution with a parameter depending on the type of rock.

One can show that under the additional assumption of stationarity the "time-reversed" process (i.e., the process observed from youngest to oldest) is also a Markov process. This means that it does not matter whether we view a sequence of lithologies in the order in which they evolved or in the order in which they are found. If one is a Markov process, so is the other. More about the theory of Markov processes can be found in Cinlar (1975), Karlin and Taylor (1975), Ross (1980), Prohorov and Rozanov (1969), and others.

STATISTICS

An essential assumption for tests of the distribution of thicknesses of rock types is the independence of the sequence of observations of each rock type. If the correctness of this assumption is not clear from the context, one should start with a test for independence of the observations. Tests for such a problem can be found in many statistics texts (e.g., Kreyszig, 1970).

A test for the underlying Markov chain was first given by Goodman (1968), and was introduced into the geological literature by Powers and Easterling (1982). They used this test to solve the problem of a Markovian structure, looking only for transitions and disregarding thicknesses of the layers. In the framework of continuous-time Markov processes, this is the same problem as to test for the underlying Markov chain.

To check whether the model of a continuous-time Markov process fits, one also has to decide if the thicknesses of the various rock types are exponentially distributed. Several statistical methods are known for this test (i.e., the Kolmogorov–Smirnoff test or the Wilks test). For details, see Kreyszig (1970), Lilliefors (1969), and Wilks (1972).

One problem that arises very often in statistics is especially serious in geology: the problem of outliers. The objects under consideration here, sedimentary lithologies, have been built up over long periods, in the range of millions of years. During this time, many catastrophic events, such as tectonism, meteorites, outbreaks of vulcanism, and so on, may have influenced the data and somewhat disturbed the Markov process. Because of this, the choice of a nonparametric procedure is crucial. The Kolmogorov–Smirnoff test is unfortunately very sensitive to outliers, and is mentioned here mainly because it is the most widely known nonparametric test.

TEST RESULTS ON CLIFTON HILL DATA

The Clifton Hill section has previously been discussed by Osborne (1971) and Ford (1967). It is a typical Cincinnatian sedimentation, Hamilton County, Ohio. It consists of shale and seven different kinds of limestone. For a classification of the limestone, see Osborne (1971).

First, a test for the Markovian structure in the underlying chain is performed. The tally matrix with the observed transitions and the expected number of transitions, computed using the iterative method described by Goodman (1968), are shown in Table 1.

An estimate for the divergence is:

$$\chi^2 = 15.7$$

From the χ^2-table critical values are

$$\chi^2(41; 975) = 24.5 > 15.7$$

and so the null-hypothesis of independence is rejected on a 5%-significance level. Goodman (1968) gives the degrees of freedom in this model as $(m - 1)^2 - m$, where m is the number of rock types. Here this gives 41 degrees of freedom.

Next, the exponentiality of the thicknesses of the rock types is tested. Table 2 shows the data for the thicknesses, the values of the Kolmogoroff–Smirnoff D-statistic, the critical values, and whether or not the hypotheses of an

Table 1

Number of transitions: 311
Observed transitions:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 37 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 25 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 59 & 10 & 39 & 23 & 7 & 10 & 4 & 0 \end{bmatrix}$$

Number of iterations: 21
Expected number of transitions under H_0:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 61 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 58 & 9 & 37 & 23 & 7 & 10 & 3 & 0 \end{bmatrix}$$

Table 2[a]

Lithology	# of Obs.	*D*-Stat	Crit. Val.	Conclusion
Limestone 1	70	0.22	0.12	Reject H_0
Limestone 2	8	0.39	0.36	Reject H_0
Limestone 3	33	0.13	0.18	Accept H_0
Limestone 4	25	0.33	0.21	Reject H_0
Limestone 5	8	0.37	0.36	Reject H_0
Limestone 6	10	0.35	0.32	Reject H_0
Limestone 7	3	0.43	0.55	Accept H_0
Shale	156	0.09	0.08	Reject H_0

[a]"Accept H_0" means that the null-hypotheses of an exponential distribution is accepted.

exponential distribution is accepted. Table 3 gives the same data for the Wilks *W*-statistic.

A further analysis was done on limestone 1 and shale in order to check the influence of outliers. Dropping the two smallest observations of limestone 1 and the two largest observations of shale leads to the results shown in Table 3 as limestone 1* and shale*. This indicates that outliers had a strong influence on the outcome of the test, and that the null hypothesis of an exponential distribution of the thicknesses of limestone 1 and of shale should be accepted.

The analysis of the Clifton Hill Section shows strong support for the hypothesis of a continuous-time Markov process. That means that the probability that layer n in the sequence is a certain lithology depends only on what the (n

Table 3[a]

Lithology	# of Obs.	*W*-Statistic	Conclusion
Limestone 1	70	0.028	Reject H_0
Limestone 2	8	0.56	Reject H_0
Limestone 3	33	0.033	Accept H_0
Limestone 4	25	0.05	Accept H_0
Limestone 5	8	0.33	Accept H_0
Limestone 6	10	0.25	Accept H_0
Limestone 7	3	0.95	Accept H_0
Shale	50	0.008	Reject H_0
Limestone 1*	68	0.0178	Accept H_0
Shale*	48	0.021	Accept H_0

[a]For limestone 1* the lowest observation and for shale* the two highest observations were dropped from the sample in order to check for outliers.

$-1)^{st}$ lithology was and not on any prior to $n - 1$. It also means that the thickness of each layer has an exponential distribution with a parameter depending on the rock type.

MARKOV RENEWAL PROCESSES

In practice, an analysis as described above will often show that the transitions from one rock type to another form a Markov chain but that some of the thicknesses are not distributed exponentially. One way of proceeding is to consider a Markov renewal process.

The stochastic process $(X, T) = ((X_n, T_n), n \in N)$ is a Markov renewal process with state space S if:

$$P(X_{n+1} = j, T_{n+1} - T_n < t | X_0, \ldots, X_n; T_0, \ldots, T_n)$$

$$= P(X_{n+1} = j, T_{n+1} - T_n < t | X_n, T_n)$$

for all $n \in N$, $j \in S$ and $t \in R^+$.

The most interesting case is when

$$Q(i, j, t) := P(X_{n+1} = j, T_{n+1} - T_n < t | X_n = i) = q_{ij} * F_i(t)$$

Thus, the transitions from one rock type to another are distributed according to a discrete-time Markov chain with transition matrix $Q = (q_{ij})$ but the thicknesses of the rock types are not neccessarily exponentially distributed but have a continuous distribution $F_i(t)$ depending on the rock type i. In this case, the stratigraphic column does not follow a continuous-time Markov process, although an analysis similar to the one described above is still possible because the transitions and the thicknesses can still be studied separately.

Continuous-time Markov processes, as considered above, are a special case of Markov renewal processes, where one has $F_i(t) = [1 - \exp(-r_i * t)]$, where $1/r_i$ is the mean thickness of rock type i.

The difficulty in applying this theory lies in finding a hypothesis for the distribution of the thicknesses. A first try could be made using a gamma distribution, because it is a generalization of the exponential distribution. Methods for estimating a distribution, so-called density estimators, are known but mathematically quite difficult and computationally rather time-consuming. Further details regarding Markov renewal processes can be found in Cinlar (1975).

CONCLUSION

Lithologies are continuously created in time, and the thicknesses of the segments can contain additional information about the structure of the lithology. This makes continuous-time Markov processes a model better fitted to describe the evolution of lithologies than discrete-time Markov chains. Statistical procedures are known and fairly easy to implement. The additional information about the distribution of the thickness of a lithology can be used in various ways and can be helpful to understand the process of sedimentation.

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