**COMBINED LOADS**

In many structures the members are required to resist more than one kind of loading (combined loading). These can often be analyzed by *superimposing the stresses and strains cause by each load acting separately*.

Superposition of stresses and strains is permissible only under the following conditions:

a. The stresses and the strains must be a linear function of the applied loads (Hooke’s law must be obeyed and the displacements must be small).

b. There must be no interaction between the various loads.

Examples: wide-flange beam supported by a cable (combined bending and axial load), cylindrical pressure vessel supported as a beam, and shaft in combined torsion and bending.
Method of Analysis:

1. Select the point on the structure where the stresses and the strains are to be determined.
2. For each load on the structure, determine the stress resultant at the cross section containing the selected point.
3. Calculate the normal and shear stresses at the selected point due to each of the stress resultant.
4. Combine the individual stresses to obtain the resultant stresses at the selected point.
5. Determine the principal stresses and maximum shear stresses at the selected point.
6. Determine the strains at the point with the aid of Hooke’s law for plane stress.
7. Select additional points and repeat the process.

\[ \sigma = \frac{P}{A} \quad \tau = \frac{T\rho}{I\rho} \quad \sigma = -\frac{My}{I} \]

\[ \tau = \frac{VQ}{lb} \quad \sigma = \frac{pr}{t} \]
The bar shown is subjected to two types of loads: a torque $T$ and a vertical load $P$.

Let us select arbitrarily two points. Point $A$ (top of the bar) and point $B$ (side of the bar - in the same cross section).

The resulting stresses acting across the section are the following:

- A twisting moment equal to the torque $T$.
- A bending moment $M$ equal to the load $P$ times the distance $b$.
- A shear force $V$ equals to the load $P$.  

**Illustration of the Method:**
The twisting moment $T$ produces a torsional shear stress:

$$\tau_{\text{torsion}} = \frac{Tr}{I_{\text{Polar}}} = \frac{2T}{\pi r^3}$$

The stress $\tau_1$ acts horizontally to the left at point $A$ and vertically downwards at point $B$.

The bending moment $M$ produces a tensile stress at point $A$:

$$\sigma_{\text{bending}} = \frac{Mr}{I} = \frac{4M}{\pi r^3}$$

However, the bending moment produces no stress at point $B$, because $B$ is located on the neutral axis.

The shear force $V$ produces no shear stress at the top of the bar (point $A$), but at point $B$ the shear stress is as follows:

$$\tau_{\text{shear}} = \frac{VQ}{Ib} = \frac{4V}{3A}$$

$\sigma_A$ and $\tau_1$ are acting in point $A$, while the $\tau_1$ and $\tau_2$ are acting in point $B$. 
Note that the element is in plane stress with 
\[ \sigma_x = \sigma_A, \; \sigma_y = 0, \; \text{and} \; \tau_{xy} = -\tau_1. \]

A stress element in point \( B \) is also in plane stress and the only stresses acting on this element are the shear stresses \( \tau_1 \) and \( \tau_2 \). Therefore
\[ \sigma_x = \sigma_y = 0 \text{ and } \tau_{xy} = -(\tau_1 + \tau_2). \]

At point \( A \): \( \sigma_x = \sigma_A, \; \sigma_y = 0, \; \text{and} \; \tau_{xy} = -\tau_1 \)

At point \( B \) \( \sigma_x = \sigma_y = 0 \) and \( \tau_{xy} = -(\tau_1 + \tau_2) \).
Of interest are the points where the stresses calculated from the flexure and shear formulas have maximum or minimum values, called critical points.

For instance, the normal stresses due to bending are largest at the cross section of maximum bending moment, which is at the support. Therefore, points $C$ and $D$ at the top and bottom of the beam at the fixed ends are critical points where the stresses should be calculated.

**Selection of Critical Areas and Points**

If the objective of the analysis is to determine the largest stresses anywhere in the structure, then the critical points should be selected at cross sections where the stress resultants have their largest values.

Furthermore, within those cross sections, the points should be selected where either the normal stresses or the shear stresses have their largest values.
Stress at which point?

point A:
\[
\begin{align*}
\sigma_x &= \sigma_{ax} = \frac{N}{A} \\
\tau_{xy} &= -\tau_{torque} - \tau_{shear} = -\frac{T_r}{J} - \frac{4F}{3A} \\
\sigma_y &= \sigma_z = \tau_{xz} = \tau_{yz} = 0
\end{align*}
\]

point B:
\[
\begin{align*}
\sigma_x &= \sigma_{ax} + \sigma_{bend} = \frac{N}{A} + \frac{Flr}{Iz} \\
\tau_{xz} &= \tau_{torque} = \frac{T_r}{J} \\
\sigma_y &= \sigma_z = \tau_{xy} = \tau_{yz} = 0
\end{align*}
\]

point C:
\[
\begin{align*}
\sigma_x &= \sigma_{ax} = \frac{N}{A} \\
\tau_{xy} &= \tau_{torque} - \tau_{shear} = \frac{T_r}{J} - \frac{4F}{3A} \\
\sigma_y &= \sigma_z = \tau_{xz} = \tau_{yz} = 0
\end{align*}
\]

point D:
\[
\begin{align*}
\sigma_x &= \sigma_{ax} - \sigma_{bend} = \frac{N}{A} - \frac{Flr}{Iz} \\
\tau_{xz} &= -\tau_{torque} = -\frac{T_r}{J} \\
\sigma_y &= \sigma_z = \tau_{xy} = \tau_{yz} = 0
\end{align*}
\]

© Thomas R. Chase, 2005
The rotor shaft of an helicopter drives the rotor blades that provide the lifting force to support the helicopter in the air. As a consequence, the shaft is subjected to a combination of torsion and axial loading.

For a 50mm diameter shaft transmitting a torque $T = 2.4kN.m$ and a tensile force $P = 125kN$, determine the maximum tensile stress, maximum compressive stress, and maximum shear stress in the shaft.

Solution

The stresses in the rotor shaft are produced by the combined action of the axial force $P$ and the torque $T$. Therefore the stresses at any point on the surface of the shaft consist of a tensile stress $\sigma_o$ and a shear stress $\tau_o$.

The tensile stress

$$\sigma = \frac{P}{A} = \frac{125kN}{\pi/4(0.05m)^2} = 63.66MPa$$

The shear stress $\tau_o$ is obtained from the torsion formula

$$\tau_{Torsion} = \frac{Tr}{I_P} = \frac{(2.4kN.m)(0.05)}{\pi(0.05)^4} = 97.78MPa$$
Knowing the stresses $\sigma_0$ and $\tau_0$, we can now obtain the principal stresses and maximum shear stresses. The principal stresses are obtained from:

$$\sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

The maximum in-plane shear stresses are obtained using the formula:

$$\sigma_{1,2} = \left(\frac{0 + 63.66}{2}\right) \pm \sqrt{\left(\frac{0 - 63.66}{2}\right)^2 + (-97.78)^2}$$

$$\sigma_1 = 135\text{MPa}$$
$$\sigma_2 = -71\text{MPa}$$

Because the principal stresses $\sigma_1$ and $\sigma_2$ have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses. Therefore, the maximum shear stress in the shaft is 103MPa.

Will it fail if $\sigma_{\text{yield}}=480\text{MPa}$?

$$\text{MSST } \Rightarrow \text{ SF } = \frac{480\text{MPa}}{103\text{MPa}} = 2.33$$

$$\sigma_{\text{VM}} = \sqrt{(135)^2 - (135)(-71) + (-71)^2} = 181.2\text{MPa}$$

$$\text{DET } \Rightarrow \text{ SF } = \frac{480\text{MPa}}{181.2\text{MPa}} = 2.65$$
A thin wall cylindrical pressure vessel with a circular cross section is subjected to internal gas pressure $p$ and simultaneously compressed by an axial load $P = 12k$. The cylinder has inner radius $r = 2.1\text{in}$. And wall thickness $t = 0.15\text{in}$. Determine the maximum allowable internal pressure $p_{allow}$ based upon an allowable shear stress of $6500\text{psi}$ in the wall of the vessel.

### Solution

The stresses on the wall of the pressure vessel are caused by a combined action of the internal pressure and the axial force. We can isolate a stress element in point $A$. The $x$-axis is parallel to the longitudinal axis of the pressure vessel and the $y$-axis is circumferential. Note that there are no shear stresses acting on the element.

The longitudinal stress $\sigma_x$ is equal to the tensile stress produced by the internal pressure minus the compressive stress produced by the axial force.

$$\sigma_x = \frac{pr}{2t} - \frac{P}{A} = \frac{pr}{2t} - \frac{P}{2\pi rt}$$
The circumferential stress $\sigma_y$ is equal to the tensile stress produced by the internal pressure. Note that $\sigma_y > \sigma_x$.

Since no shear stresses act on the element the above stresses are also the principal stresses

Substituting numerical values

$$
\sigma_1 = \frac{pr}{t} = \frac{p(2.1\text{in})}{0.15\text{in}} = 14.0\ p
$$

$$
\sigma_2 = \frac{pr}{2t} - \frac{P}{2\pi rt} = \frac{p(2.1\text{in})}{2(0.15\text{in})} - \frac{12k}{[2\pi(2.1\text{in})(0.15\text{in})]} = 7.0\ p - 6063\ \text{psi}
$$

**In-Plane Shear Stresses**

The maximum in-plane shear stress is

$$
\tau_{\text{Max}} = \frac{(\sigma_1 - \sigma_2)}{2} = \frac{((14.0\ \text{p})-(7.0\ p - 6063\ \text{psi}))}{2} = 3.5\ p + 3032\ \text{psi}
$$

Since $\tau_{\text{max}}$ is limited to 6500psi then

$$
65000\ \text{psi} = 3.4\ p + 3032\ \text{psi} \Rightarrow p_{\text{allowed}} = 990\ \text{psi}
$$
Out-of-Plane Shear Stresses

The maximum out-of-plane shear stress is either

From the first equation we get

\[ 65000 \text{ psi} = 3.5 p - 3032 \text{ psi} \]
\[ p_{\text{allowed}} = 2720 \text{ psi} \]

From the second equation we get:

\[ 65000 \text{ psi} = 7.0 p \Rightarrow p_{\text{allowed}} = 928 \text{ psi} \]

Allowable internal pressure

Comparing the three calculated values for the allowable pressure, we see that \((p_{\text{allow}})^3 = 928\text{psi} \) governs.

At this pressure, the principal stresses are

\[ \sigma_1 = 13000\text{psi} \quad \text{and} \]
\[ \sigma_2 = 430\text{psi}. \]

These stresses have the same signs, thus confirming that one of the out-of-plane shear stresses must be the largest shear stress.
A sign of dimensions 2.0m x 1.2m is supported by a hollow circular pole having outer diameter 220mm and inner diameter 180mm (see figure). The sign offset 0.5m from the centerline of the pole and its lower edge is 6.0m above the ground.

Determine the principal stresses and maximum shear stresses at points A and B at the base of the pole due to wind pressure of 2.0kPa against the sign.

**Solution**

**Stress Resultant:** The wind pressure against the sign produces a resultant force $W$ that acts at the midpoint of the sign and it is equal to the pressure $p$ times the area $A$ over which it acts:

$$W = pA = (2.0\text{kPa})(2.0\text{m} \times 1.2\text{m}) = 4.8\text{kN}$$

The line of action of this force is at height $h = 6.6\text{m}$ above the ground and at distance $b = 1.5\text{m}$ from the centerline of the pole.

The wind force acting on the sign is statically equivalent to a lateral force $W$ and a torque $T$ acting on the pole.
The torque is equal to the force $W$ times the distance $b$:

$$ T = Wb = (4.8\text{kN})(1.5\text{m}) $$

$$ T = 7.2\text{kN} \cdot \text{m} $$

The stress resultant at the base of the pole consists of a bending moment $M$, a torque $T$ and a shear force $V$. Their magnitudes are:

$M = Wh = (4.8\text{kN})(6.6\text{m}) = 31.68\text{kN} \cdot \text{m}$

$T = 7.2\text{kN} \cdot \text{m}$

$V = W = 4.8\text{kN}$

Examination of these stress resultants shows that maximum bending stresses occur at point $A$ and maximum shear stresses at point $B$.

Therefore, $A$ and $B$ are critical points where the stresses should be determined.

**Stresses at points $A$ and $B$**

The bending moment $M$ produces a tensile stress $\sigma_a$ at point $A$, but no stress at point $B$ (which is located on the neutral axis).
The torque \( T \) produces shear stresses \( \tau \) at points \( A \) and \( B \).

Finally, we need to calculate the direct shear stresses at points \( A \) and \( B \) due to the shear force \( V \).
The shear stress at point $A$ is zero, and the shear stress at point $B$ ($\tau_2$) is obtained from the shear formula for a circular tube

$$\tau_{2,\text{Max}} = \frac{2V}{A} = \frac{2(4800)}{0.01257\text{m}^2} = 0.7637\text{MPa}$$

The stresses acting on the cross section at points $A$ and $B$ have now been calculated.

\[\tau_2 = \frac{VQ}{Ib}\]
\[I = \left[\frac{\pi(d_2^4 - d_1^4)}{64}\right]\]
\[Q = \frac{2}{3}(r_3^3 - r_1^3)\]
\[b = 2(r_2 - r_1)\]
Stress Elements
For both elements the y-axis is parallel to the longitudinal axis of the pole and the x-axis is horizontal.

Point A:
\[ \sigma_x = 0 \]
\[ \sigma_y = \sigma_a = 54.91 \text{MPa} \]
\[ \tau_{xy} = \tau_1 = 6.24 \text{MPa} \]

Principal stresses at Point A

\[
\sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}
\]

Substituting \( \sigma_{1,2} = 27.5 \text{MPa} \pm 28.2 \text{MPa} \)
\[ \sigma_1 = 55.7 \text{MPa} \quad \text{and} \quad \sigma_2 = -0.7 \text{MPa} \]

The maximum in-plane shear stresses can be obtained from the equation

\[
\tau_{\text{MAX}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = 28.2 \text{MPa}
\]

Because the principal stresses have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses. Then, \( \tau_{\text{max}} = 28.2 \text{MPa} \).
**Point B:**

\[ \sigma_x = \sigma_y = 0 \]

\[ \tau_{xy} = \tau_1 + \tau_2 \]

\[ \tau_{xy} = 6.24 \text{MPa} + 0.76 \text{MPa} = 7.0 \text{MPa} \]

Principal stresses at point \textit{B} are

\[ \sigma_1 = 7.0 \text{MPa} \quad \sigma_2 = -7.0 \text{ MPa} \]

And the maximum in-plane shear stress is

\[ \tau_{\text{max}} = 7.0 \text{MPa} \]

The maximum out-of-plane shear stresses are half of this value.

**Note**

If the largest stresses anywhere in the pole are needed, then we must also determine the stresses at the critical point diametrically opposite point \textit{A}, because at that point the compressive stress due to bending has its largest value.

The principal stresses at that point are

\[ \sigma_1 = 0.7 \text{MPa} \quad \text{and} \quad \sigma_2 = -55.7 \text{MPa} \]

The maximum shear stress is \textit{28.2MPa}.

(In this analysis only the effects of wind pressure are considered. Other loads, such as weight of the structure, also produce stresses at the base of the pole).
A tubular post of square cross section supports a horizontal platform (see figure). The tube has outer dimension $b = 6\text{in.}$ And wall thickness $t = 0.5\text{in.}$ The platform has dimensions $6.75\text{in} \times 24.0\text{in}$ and supports an uniformly distributed load of $20\text{psi}$ acting over its upper surface. The resultant of this distributed load is a vertical force $P_1 = (20\text{psi})(6.75\text{in} \times 24.0\text{in}) = 3240\text{lb}$ This force acts at the midpoint of the platform, which is at distance $d = 9\text{in.}$ from the longitudinal axis of the post. A second load $P_2 = 800\text{lb}$ acts horizontally on the post at height $h = 52\text{in}$ above the base. Determine the principal stresses and maximum shear stresses at points A and B at the base of the post due to the loads $P_1$ and $P_2$. 

Example
Solution

Stress Resultants
The force $P_1$ acting on the platform is statically equivalent to a force $P_1$ and a moment $M_1 = P_1d$ acting on the centroid of the cross section of the post.

The load $P_2$ is also shown.

The stress resultant at the base of the post due to the loads $P_1$ and $P_2$ and the moment $M_1$ are as follows:

(A) An axial compressive force $P_1 = 3240\text{lb}$

(B) A bending moment $M_1$ produced by the force $P_1$:
$$M_1 = P_1d = (3240\text{lb})(9\text{in}) = 29160\text{lb-in}$$

(C) A shear force $P_2 = 800\text{lb}$

(D) A bending moment $M_2$ produced by the force $P_2$:
$$M_2 = P_2h = (800\text{lb})(52\text{in}) = 41600\text{lb-in}$$

Examinations of these stress resultants shows that both $M_1$ and $M_2$ produce maximum compressive stresses at point A and the shear force produces maximum shear stresses at point B. Therefore, A and B are the critical points where the stresses should be determined.
Stresses at points A and B

(A) The axial force \( P_1 \) produces uniform compressive stresses throughout the post. These stresses are \( \sigma_{P1} = \frac{P_1}{A} \) where \( A \) is the cross section area of the post. 

\[
A = b^2 - (b - 2t)^2 = 4t(b-t) = 4 (0.5\text{in})(6\text{in} - 0.5\text{in}) = 11.0\text{in}^2
\]

\[
\sigma_{P1} = \frac{P_1}{A} = \frac{3240\text{lb}}{11.00\text{in}^2} = 295\text{psi}
\]

(B) The bending moment \( M_1 \) produces compressive stresses \( \sigma_{M1} \) at points A and B. These stresses are obtained from the flexure formula:

\[
\sigma_{M1} = \frac{M_1 (b / 2)}{I}
\]

where \( I \) is the moment of inertia of the cross section. The moment of inertia is:

\[
I = \frac{b^4 - (b - 2t)^4}{12} = \frac{[6\text{in}]^4 - (5\text{in})^4}{12} = 55.92\text{in}^4
\]

Thus,

\[
\sigma_{M1} = \frac{M_1 b}{2I} = \frac{29160\text{lb.in})(6\text{in})}{(2)(55.92\text{in}^4)} = 1564\text{psi}
\]
(C) The shear force $P_2$ produces a shear stress at point $B$ but not at point $A$. We know that an approximate value of the shear stress can be obtained by dividing the shear force by the web area.

$$\tau_{P_2} = \frac{P_2}{A_{\text{web}}} = \frac{P_2}{2t(b-2t)} = \frac{800\text{lb}}{(2)(0.5\text{in})(6\text{in}-1\text{in})} = 160\text{psi}$$

The stress $\tau_{P_2}$ acts at point $B$ in the direction shown in the above figure. We can calculate the shear stress $\tau_{P_2}$ from the more accurate formula. The result of this calculation is $\tau_{P_2} = 163\text{psi}$, which shows that the shear stress obtained from the approximate formula is satisfactory.

D) The bending moment $M_2$ produces a compressive stress at point $A$ but no stress at point $B$. The stress at $A$ is

$$\sigma_{M_2} = \frac{M_2 b}{2I} = \frac{(41600\text{lb.in})(6\text{in})}{(2)(55.92\text{in}^4)} = 2232\text{psi}.$$ 

This stress is also shown in the above figure.
**Stress Elements**

Each element is oriented so that the $y$-axis is vertical (i.e. parallel to the longitudinal axis of the post) and the $x$-axis is horizontal axis.

**Point A:** The only stress in point $A$ is a compressive stress $\sigma_a$ in the $y$ direction.

\[
\sigma_a = \sigma_{P1} + \sigma_{M1} + \sigma_{M2}
\]

\[
\sigma_a = 295\text{psi} + 1564\text{psi} + 2232\text{psi} = 4090\text{psi}
\]

Thus, this element is in uniaxial stress.

**Principal Stresses and Maximum Shear Stress**

\[
\sigma_x = 0
\]

\[
\sigma_y = -\sigma_a = -4090\text{psi}
\]

\[
\tau_{xy} = 0
\]

Since the element is in uniaxial stress,

\[
\sigma_1 = \sigma_x \text{ and } \sigma_2 = \sigma_y = -4090\text{psi}
\]

And the maximum in-plane shear stress is

\[
\tau_{\text{max}} = (\sigma_1 - \sigma_2) / 2 = \frac{1}{2} (4090\text{psi}) = 2050\text{psi}
\]

The maximum out-of-plane shear stress has the same magnitude.
Point B:
Here the compressive stress in the $y$ direction is
\[
\sigma_B = \sigma_{P1} + \sigma_{M1}
\]
\[
\sigma_B = 295\text{psi} + 1564\text{psi} = 1860\text{psi} \text{ (compression)}
\]

And the shear stress is
\[
\tau_B = \tau_{P2} = 160\text{psi}
\]

The shear stress acts leftward on the top face and downward on the $x$ face of the element.

**Principal Stresses and Maximum Shear Stress**

\[
\sigma_x = 0
\]
\[
\sigma_y = -\sigma_B = -1860\text{psi}
\]
\[
\tau_{xy} = -\tau_{P2} = -160\text{psi}
\]

Substituting $\sigma_{1,2} = -930\text{psi} +/- 944\text{psi}$

$\sigma_1 = 14\text{psi}$ and $\sigma_2 = -1870\text{psi}$

The maximum in-plane shear stresses can be obtained from the equation

Because the principal stresses have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses.

Then, $\tau_{max} = 944\text{psi}$. 

\[
\tau_{MAX} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + (\tau_{xy})^2} = 944 \text{ psi}
\]
Three forces are applied to a short steel post as shown. Determine the principle stresses, principal planes and maximum shearing stress at point $H$.

**Solution**

*Determine internal forces in Section EFG.*

- $V_x = -30 \text{kN}$
- $P = 50 \text{kN}$
- $V_z = -75 \text{kN}$
- $M_x = (50 \text{kN})(0.130 \text{m}) - (75 \text{kN})(0.200 \text{m})$
- $M_x = -8.5 \text{kN} \cdot \text{m}$
- $M_y = 0$
- $M_z = (30 \text{kN})(0.100 \text{m}) = 3 \text{kN} \cdot \text{m}$
Note: Section properties,

**Evaluate the stresses at H.**

\[
A = (0.040 \text{ m})(0.140 \text{ m}) = 5.6 \times 10^{-3} \text{ m}^2
\]

\[
I_x = \frac{1}{12} (0.040 \text{ m})(0.140 \text{ m})^3 = 9.15 \times 10^{-6} \text{ m}^4
\]

\[
I_z = \frac{1}{12} (0.140 \text{ m})(0.040 \text{ m})^3 = 0.747 \times 10^{-6} \text{ m}^4
\]

**Normal stress at H.**

\[
\sigma_y = \frac{P}{A} + \frac{|M_z|}{I_z} a - \frac{|M_x|}{I_x} b
\]

\[
= \frac{50 \text{ kN}}{5.6 \times 10^{-3} \text{ m}^2} + \frac{(3 \text{ kN} \cdot \text{m})(0.020 \text{ m})}{0.747 \times 10^{-6} \text{ m}^4} - \frac{(8.5 \text{ kN} \cdot \text{m})(0.025 \text{ m})}{9.15 \times 10^{-6} \text{ m}^4}
\]

\[
= (8.93 + 80.3 - 23.2) \text{ MPa} = 66.0 \text{ MPa}
\]

**Shear stress at H.**

\[
Q = A \bar{y} = [(0.040 \text{ m})(0.045 \text{ m})](0.0475 \text{ m})
\]

\[
= 85.5 \times 10^{-6} \text{ m}^3
\]

\[
\tau_{yz} = \frac{V_z}{I_x t} = \frac{(75 \text{ kN})(85.5 \times 10^{-6} \text{ m}^3)}{(9.15 \times 10^{-6} \text{ m}^4)(0.040 \text{ m})}
\]

\[
= 17.52 \text{ MPa}
\]
Calculate principal stresses and maximum shearing stress.

\[ \tau_{\text{max}} = R = \sqrt{33.0^2 + 17.52^2} = 37.4 \text{ MPa} \]

\[ \sigma_{\text{max}} = OC + R = 33.0 + 37.4 = 70.4 \text{ MPa} \]

\[ \sigma_{\text{min}} = OC - R = 33.0 - 37.4 = -7.4 \text{ MPa} \]

\[ \tan 2\theta_p = \frac{CY}{CD} = \frac{17.52}{33.0} \quad 2\theta_p = 27.96^\circ \]

\[ \theta_p = 13.98^\circ \]

\[ \tau_{\text{max}} = 37.4 \text{ MPa} \]

\[ \sigma_{\text{max}} = 70.4 \text{ MPa} \]

\[ \sigma_{\text{min}} = -7.4 \text{ MPa} \]

\[ \theta_p = 13.98^\circ \]
The cantilever tube shown is to be made of 2014 aluminum alloy treated to obtain a specified minimum yield strength of 276MPa. We wish to select a stock size tube (according to the table below). Using a design factor of $n=4$.

The bending load is $F=1.75kN$, the axial tension is $P=9.0kN$ and the torsion is $T=72N.m$. What is the realized factor of safety?

*Consider the critical area (top surface).*
\[ \sigma_{VM} \leq \frac{S_y}{n} = \frac{0.276}{4} \text{ GPa} = 0.0690 \text{ GPa} \]

\[ \sigma_x = \frac{P}{A} + \frac{Mc}{I} \]

Maximum bending moment = 120F

\[ \sigma_x = \frac{9kN}{A} + \frac{120mm \times 1.75kN \times \left( \frac{d}{2} \right)}{I} \]

\[ \tau_{zx} = \frac{Tr}{J} = \frac{72 \times \left( \frac{d}{2} \right)}{J} = \frac{36d}{J} \]

\[ \sigma_{VM} = \left( \sigma_x^2 + 3\tau_{zx}^2 \right)^{\frac{1}{2}} \]

For the dimensions of that tube

\[ n = \frac{S_y}{\sigma_{VM}} = \frac{0.276}{0.06043} = 4.57 \]

---

### Table

<table>
<thead>
<tr>
<th>Size, in</th>
<th>( w_o )</th>
<th>( w_s )</th>
<th>( A )</th>
<th>( I )</th>
<th>( k )</th>
<th>( Z )</th>
<th>( J )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \times \frac{1}{8} )</td>
<td>0.416</td>
<td>1.128</td>
<td>0.344</td>
<td>0.034</td>
<td>0.313</td>
<td>0.067</td>
<td>0.067</td>
</tr>
<tr>
<td>( 1 \times \frac{1}{4} )</td>
<td>0.713</td>
<td>2.003</td>
<td>0.589</td>
<td>0.046</td>
<td>0.280</td>
<td>0.092</td>
<td>0.092</td>
</tr>
<tr>
<td>( 1 \frac{1}{2} \times \frac{1}{8} )</td>
<td>0.653</td>
<td>1.769</td>
<td>0.540</td>
<td>0.129</td>
<td>0.488</td>
<td>0.172</td>
<td>0.257</td>
</tr>
<tr>
<td>( 1 \frac{1}{2} \times \frac{1}{4} )</td>
<td>1.188</td>
<td>3.338</td>
<td>0.982</td>
<td>0.199</td>
<td>0.451</td>
<td>0.266</td>
<td>0.399</td>
</tr>
<tr>
<td>( 2 \times \frac{1}{8} )</td>
<td>0.891</td>
<td>2.670</td>
<td>0.736</td>
<td>0.325</td>
<td>0.664</td>
<td>0.325</td>
<td>0.650</td>
</tr>
<tr>
<td>( 2 \times \frac{1}{4} )</td>
<td>1.663</td>
<td>4.673</td>
<td>1.374</td>
<td>0.537</td>
<td>0.625</td>
<td>0.537</td>
<td>1.074</td>
</tr>
<tr>
<td>( 2 \frac{1}{2} \times \frac{1}{8} )</td>
<td>1.129</td>
<td>3.050</td>
<td>0.933</td>
<td>0.660</td>
<td>0.841</td>
<td>0.528</td>
<td>1.319</td>
</tr>
<tr>
<td>( 2 \frac{1}{2} \times \frac{1}{4} )</td>
<td>2.138</td>
<td>6.008</td>
<td>1.767</td>
<td>1.132</td>
<td>0.800</td>
<td>0.906</td>
<td>2.276</td>
</tr>
<tr>
<td>( 3 \times \frac{1}{4} )</td>
<td>2.614</td>
<td>7.343</td>
<td>2.160</td>
<td>2.059</td>
<td>0.976</td>
<td>1.373</td>
<td>4.117</td>
</tr>
<tr>
<td>( 3 \times \frac{3}{8} )</td>
<td>3.742</td>
<td>10.51</td>
<td>3.093</td>
<td>2.718</td>
<td>0.938</td>
<td>1.812</td>
<td>5.436</td>
</tr>
<tr>
<td>( 4 \times \frac{3}{16} )</td>
<td>2.717</td>
<td>7.654</td>
<td>2.246</td>
<td>4.090</td>
<td>1.350</td>
<td>2.045</td>
<td>8.180</td>
</tr>
<tr>
<td>( 4 \times \frac{3}{8} )</td>
<td>5.167</td>
<td>14.52</td>
<td>4.271</td>
<td>7.090</td>
<td>1.289</td>
<td>3.544</td>
<td>14.180</td>
</tr>
</tbody>
</table>

### Table (Continued)

<table>
<thead>
<tr>
<th>Size, mm</th>
<th>( m )</th>
<th>( A )</th>
<th>( l )</th>
<th>( k )</th>
<th>( Z )</th>
<th>( J )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 12 \times 2 )</td>
<td>0.490</td>
<td>0.628</td>
<td>0.082</td>
<td>0.361</td>
<td>0.136</td>
<td>0.163</td>
</tr>
<tr>
<td>( 16 \times 2 )</td>
<td>0.687</td>
<td>0.879</td>
<td>0.220</td>
<td>0.500</td>
<td>0.275</td>
<td>0.440</td>
</tr>
<tr>
<td>( 16 \times 3 )</td>
<td>0.956</td>
<td>1.225</td>
<td>0.273</td>
<td>0.472</td>
<td>0.341</td>
<td>0.545</td>
</tr>
<tr>
<td>( 20 \times 4 )</td>
<td>1.569</td>
<td>2.010</td>
<td>0.684</td>
<td>0.583</td>
<td>0.684</td>
<td>1.367</td>
</tr>
<tr>
<td>( 25 \times 4 )</td>
<td>2.060</td>
<td>2.638</td>
<td>1.508</td>
<td>0.756</td>
<td>1.206</td>
<td>3.015</td>
</tr>
<tr>
<td>( 25 \times 5 )</td>
<td>2.452</td>
<td>3.140</td>
<td>1.669</td>
<td>0.729</td>
<td>1.336</td>
<td>3.338</td>
</tr>
<tr>
<td>( 30 \times 4 )</td>
<td>2.550</td>
<td>3.266</td>
<td>2.827</td>
<td>0.930</td>
<td>1.885</td>
<td>5.652</td>
</tr>
<tr>
<td>( 30 \times 5 )</td>
<td>3.065</td>
<td>3.925</td>
<td>3.192</td>
<td>0.901</td>
<td>2.128</td>
<td>6.381</td>
</tr>
<tr>
<td>( 42 \times 4 )</td>
<td>3.727</td>
<td>4.773</td>
<td>8.717</td>
<td>1.351</td>
<td>4.151</td>
<td>17.430</td>
</tr>
<tr>
<td>( 42 \times 5 )</td>
<td>4.536</td>
<td>5.809</td>
<td>10.130</td>
<td>1.320</td>
<td>4.825</td>
<td>20.255</td>
</tr>
<tr>
<td>( 50 \times 4 )</td>
<td>4.512</td>
<td>5.778</td>
<td>15.409</td>
<td>1.632</td>
<td>6.164</td>
<td>30.810</td>
</tr>
<tr>
<td>( 50 \times 5 )</td>
<td>5.517</td>
<td>7.065</td>
<td>18.118</td>
<td>1.601</td>
<td>7.247</td>
<td>36.226</td>
</tr>
</tbody>
</table>
A certain force $F$ is applied at $D$ near the end of the 15-in lever, which is similar to a socket wrench. The bar $OABC$ is made of $AISI 1035$ steel, forged and heat treated so that it has a minimum (ASTM) yield strength of $81kpsi$. Find the force (F) required to initiate yielding. Assume that the lever $DC$ will not yield and that there is no stress concentration at $A$.

Solution:

1) Find the critical section

The critical sections will be either point $A$ or Point $O$. As the moment of inertia varies with $r^4$ then point $A$ in the 1in diameter is the weakest section.
2) Determine the stresses at the critical section

\[
\sigma_x = \frac{My}{I} = \frac{M \left( \frac{d}{2} \right)}{\pi d^4} = \frac{32 \times F \times 14 \text{in}}{\pi d^3} = 142.6F
\]

3) Chose the failure criteria.

The AISI 1035 is a ductile material. Hence, we need to employ the distortion-energy theory.

\[
\tau_{zx} = \frac{Tr}{J} = \frac{T \left( \frac{d}{2} \right)}{\pi d^4} = \frac{16 \times F \times 15 \text{in}}{\pi (1 \text{in})^3} = 76.4F
\]

\[
\sigma_{VM} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau_{xy}^2} = \sqrt{\sigma_x^2 + 3 \tau_{zx}^2} = 194.5F
\]

\[
F = \frac{S_y}{\sigma_{VM}} = \frac{81000}{194.5} = 416 \text{lbf}
\]
Apply the MSS theory. For a point undergoing plane stress with only one non-zero normal stress and one shear stress, the two non-zero principal stresses ($\sigma_A$ and $\sigma_B$) will have opposite signs (Case 2).

\[
\tau_{\text{max}} = \frac{\sigma_A - \sigma_B}{2} = \frac{S_y}{2} = \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{zx}^2}
\]

\[
\sigma_A - \sigma_B \geq S_y = 2\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{zx}^2} = \sqrt{\sigma_x^2 + 4\tau_{zx}^2}
\]

\[
81000 = \left((142.6F)^2 + 4 \times (76.4F)^2\right)^{1/2}
\]

$F = 388\text{lbf}$
Example

A round cantilever bar is subjected to torsion plus a transverse load at the free end. The bar is made of a ductile material having a yield strength of \textbf{50000psi}. The transverse force \((P)\) is \textbf{500lb} and the torque is \textbf{1000lb-in} applied to the free end. The bar is \textbf{5in long (L)} and a safety factor of \textbf{2} is assumed. Transverse shear can be neglected. Determine the minimum diameter to avoid yielding using both MSS and DET criteria.

Solution

1) Determine the critical section

The critical section occurs at the wall.
\[ \sigma_x = \frac{Mc}{I} = \frac{PL\left(\frac{d}{2}\right)}{\pi d^4} = \frac{32PL}{\pi d^3} \]

\[ \tau_{xy} = \frac{T_c}{J} = \frac{T\left(\frac{d}{2}\right)}{\pi d^4} = \frac{16T}{\pi d^3} \]

\[ \sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} = \left(\frac{\sigma_x}{2}\right) \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + (\tau_{xy})^2} \]

\[ \sigma_{1,2} = \frac{16PL}{\pi d^3} \pm \sqrt{\left(\frac{16PL}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2} = \frac{16}{\pi d^3} \left[ PL \pm \sqrt{(PL)^2 + T^2} \right] \]

\[ \sigma_{1,2} = \frac{16}{\pi d^3} \left[ 500 \times 5 \pm \sqrt{(500 \times 5)^2 + 1000^2} \right] \]
The stresses are in the wrong order. Rearranged to

\[
\begin{align*}
\sigma_1 &= \frac{26450}{d^3} \\
\sigma_2 &= -\frac{980.8}{d^3} \\
\sigma_3 &= -\frac{980.8}{d^3}
\end{align*}
\]

\[
\tau_{MAX} = \frac{\sigma_1 - \sigma_3}{2} = \frac{26450 - (-980.8)}{2d^3} = \frac{13715.4}{d^3}
\]

\[
\sigma_1 - \sigma_3 = 2\tau_{MAX} \leq \frac{S_y}{n} = \frac{50000}{2} = 25000
\]

\[
d \geq 1.031 \text{ in}
\]

\[
\sigma_{VM} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} = \sqrt{\left(\frac{26450}{d^3}\right)^2 + \left(\frac{-980.8}{d^3}\right)^2 - \left(\frac{26450}{d^3}\right)\left(\frac{-980.8}{d^3}\right)}
\]

\[
\sigma_{VM} = \frac{26950}{d^3} \leq \frac{S_y}{n} = \frac{50000}{2}
\]

\[
d \geq 1.025 \text{ in}
\]
Example

The factor of safety for a machine element depends on the particular point selected for the analysis. Based upon the DET theory, determine the safety factor for points A and B.

This bar is made of AISI 1006 cold-drawn steel (S_y = 280 MPa) and it is loaded by the forces F = 0.55 kN, P = 8.0 kN and T = 30 N.m

Solution:

Point A

\[
\sigma_x = \frac{Mc}{I} + \frac{P}{\text{Area}} = \frac{Fl\left(\frac{d}{2}\right)}{\pi d^4} + \frac{P}{\pi d^2} = \frac{32Fl}{\pi d^3} + \frac{4P}{\pi d^2}
\]

\[
\sigma_x = \frac{32(0.55)(10^3)(0.1)}{\pi(0.02)^3} + \frac{4(8)(10^3)}{\pi(0.02)^2} = 95.49 \text{ MPa}
\]
\[ \tau_{xy} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(30)}{\pi (0.020)^3} = 19.10 \text{MPa} \]

\[ \sigma_{VM} = \sqrt{\left(\sigma_x^2 + 3\tau_{xy}^2\right)} = \left[95.49^2 + 3(19.1)^2\right]^{1/2} = 101.1 \text{MPa} \]

\[ n = \frac{S_y}{\sigma_{VM}} = \frac{280}{101.1} = 2.77 \]

Point B

\[ \sigma_x = \frac{4P}{\pi d^2} = \frac{4(8)(10^3)}{\pi (0.02)^2} = 25.47 \text{MPa} \]

\[ \tau_{xy} = \frac{16T}{\pi d^3} + \frac{4V}{3A} = \frac{16(30)}{\pi (0.02)^3} + \frac{4(0.55)(10^3)}{3\left(\frac{\pi}{4}\right)(0.02)^2} = 21.43 \text{MPa} \]

\[ \sigma_{VM} = \left[25.47^2 + 3(21.43)^2\right]^{1/2} = 45.02 \text{MPa} \]

\[ n = \frac{280}{45.02} = 6.22 \]
The shaft shown in the figure below is supported by two bearings and carries two V-belt sheaves. The tensions in the belts exert horizontal forces on the shaft, tending to bend it in the x-z plane. Sheaves B exerts a clockwise torque on the shaft when viewed towards the origin of the coordinate system along the x-axis. Sheaves C exerts an equal but opposite torque on the shaft. For the loading conditions shown, determine the principal stresses and the safety factor on the element K, located on the surface of the shaft (on the positive z-side), just to the right of sheave B. Consider that the shaft is made of a steel of a yield strength of 81ksi.
Shaft dia. = 1.25 in
T = Torque = 11000 lb \cdot in

(a) Pictorial view of shaft

(b) Forces acting on shaft at B and C caused by belt drives
Shearing force = 165lb

Bending Moment = -1540lb-in

\[ \sigma_x = -\frac{Mc}{I} = -\frac{M(r)}{\pi r^4} = -4 \frac{-1540}{\pi (0.625)^3} = 8.031 ksi \]

\[ \tau_{xz} = \frac{Tr}{J} = \frac{2T}{\pi r^3} = \frac{2(1100)}{\pi (0.625)^3} = 2.868 ksi \]


\[
\sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = \left(\frac{\sigma_x}{2}\right) \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \left(\tau_{xy}\right)^2}
\]

\[
\sigma_{1,2} = \frac{8.03}{2} \pm \sqrt{\left(\frac{8.03}{2}\right)^2 + (2.868)^2}
\]

\[
\sigma_1 = 8.95 ksi \quad \sigma_2 = -0.92 ksi
\]

\[
MSS \ldots \tau_{Max} = \sigma_1 - \sigma_3 = \frac{8.95 - (-0.92)}{2} = 4.935 ksi
\]

\[
Safety\,.\,Factor = n = \frac{S_y}{\tau_{Max}} = \frac{81}{4.935} = 8.2
\]

\[
DET \ldots \sigma_{VM} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3} = \sqrt{(8.95)^2 + (-0.92)^2 - (8.95)(-0.92)} = 9.44 ksi
\]

\[
Safety\,.\,Factor = n = \frac{S_y}{\sigma_{VM}} = \frac{81}{9.44} = 8.58
\]
A horizontal bracket $ABC$ consists of two perpendicular arms $AB$ and $BC$, of $1.2m$ and $0.4m$ in length respectively. The Arm $AB$ has a solid circular cross section with diameter equal to $60mm$. At point $C$ a load $P_1=2.02kN$ acts vertically and a load $P_2=3.07kN$ acts horizontally and parallel to arm $AB$. For the points $p$ and $q$, located at support $A$, calculate:

(1) The principal stresses.

(2) the maximum in-plane shear stress.