Stress Concentration

Stresses at or near a discontinuity such as a hole in a plate are higher than if the discontinuity does not exist. Elementary stress equations do not apply in stress concentrations.

\[
\sigma_{Max} = K_t \times \sigma_{avg}
\]

Where \( K_t \) is the stress concentration factor.
Stress concentration occurs at transition of cross sections. The more abrupt the transition, the higher are the stress concentrations.

**Stress Concentration Factor \( K_t \)**

\( K_t \) is difficult to calculate and it is usually determined by some experimental technique (photoelastic analysis of a plastic model or by numerical simulation of the stress field).

The values of \( K_t \) can be found published in Charts.  
The values of \( K_t \) are geometric properties.  
\( K_t \) is very important in brittle materials.  
In ductile materials, \( K_t \) is very important in fatigue calculations. I must be taken into account if safety is critical.
**Photoelasticity:** Photoelasticity is a visual method for viewing the full field stress distribution in a photoelastic material. When a photoelastic material is strained and viewed with a polariscope, distinctive colored fringe patterns are seen. Interpretation of the pattern reveals the overall strain distribution.

**Radiometric Thermoelasticity:** When materials are stressed the change in atomic spacing creates temperature differences in the material. Cameras which sense differences in temperature can be used to display the stress field in special materials.
Effect of Geometry

The discontinuity geometry has a significant effect on the stress distribution around it.

\[
\begin{bmatrix}
\sigma_x & -1 \\
1 & \sigma_y
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
\]

\[
\sigma_{Max} = \left[1 + 2\left(\frac{a}{b}\right)^{\frac{1}{2}}\right]\sigma_{Nominal}
\]

The theoretical stress concentration at the edge of the hole is:

As \(b\) approaches zero, the situation approaches that of a very fine crack. The stress at the edge become very large. The size and orientation of the crack with respect to the applied stresses play a very large role.
Stress Concentration Factors for Round Bar with fillet

Tension

Bending

Spotts, Fig. 2-8, Peterson

Spotts, Fig. 2-9, Peterson
Torsion

Round bar with a groove

Tension
Stress concentration for a rectangular plate with fillet

Tension

Bending

Figure 6.3 Stress concentration factor for rectangular plate with fillet. (a) Axial Load. [Adapted from Collins (1981).]
Stress concentration for a plate with a hole

Tension

Bending

\[
\sigma_{\text{swg}} = \frac{P}{A} = \frac{P}{(b-d)h}
\]
Example 1:

Given a flat plate of a brittle material with a major height \((H)\) of 4.5 in, a minor height \((h)\) of 2.5 in, a fillet radius \((r)\) of 0.5 in and a width \((b)\) of 1 in. Find the stress concentration factor and the maximum stress for the following conditions: (a) axial loading; and (b) pure bending.

Solution

\[
\frac{H}{h} = \frac{4.5}{2.5} = 1.80 \quad \frac{r}{h} = \frac{0.5}{2.5} = 0.2
\]

From the figure

\[
K_t = 1.8 \quad \sigma_{Max} = 1.8 \frac{P}{2.5 \times 1} = 0.72P
\]

From the figure

\[
K_t = 1.5 \quad \sigma_{Max} = 1.44M
\]
Example 2:
A 50mm wide, 5mm high rectangular plate has a 5mm diameter central hole. The allowable stress due to applying a tensile force is 700MPa. Find (a) the maximum tensile force that can be applied; (b) the maximum bending moment that can be applied; (c) the maximum tensile force and bending moment if the hole if there is no-hole. Compare results.

Solution
\[
\frac{d}{b} = \frac{5}{50} = 0.1 \\
Area = A = (b - d)h = 0.225 \times 10^{-3} \ m^2
\]

From the Figure
\[
K_t = 2.70 \\
P_{\text{Max}} = \frac{\sigma_{\text{Allowable}} A}{K_t} = 58.33 \ kN
\]

Without a hole
\[
Area = bh = 0.25 \times 10^{-3} \ m^2 \\
P_{\text{Max}} = \sigma_{\text{Allowable}} \times A = 175 \ kN
\]
\[
\frac{d}{b} = 0.1 \quad \frac{d}{h} = 1 \quad K_t = 2.04 \quad M_{\text{Max}} = \frac{Ah\sigma_{\text{Allow}}}{6K_t} = 64.34 \text{N.m}
\]

Without a hole \[
M_{\text{Max}} = \frac{\sigma_{\text{Allow}}bh^2}{6} = 145.8 \text{N.m}
\]
Example 3: Determine the largest axial load $P$ that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64b.
  \[
  \frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.50 \quad \frac{r}{d} = \frac{8 \text{ mm}}{40 \text{ mm}} = 0.20
  \]
  \[K_t = 1.82\]

- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.
  \[
  \sigma_{\text{ave}} = \frac{\sigma_{\text{max}}}{K_t} = \frac{165 \text{ MPa}}{1.82} = 90.7 \text{ MPa}
  \]

- Apply the definition of normal stress to find the allowable load.
  \[
  P = A\sigma_{\text{ave}} = (40 \text{ mm})(10 \text{ mm})(90.7 \text{ MPa})
  = 36.3 \times 10^3 \text{ N}
  \]
  \[P = 36.3 \text{ kN}\]
Example:
The cylindrical bar is made of AISI 1006 hot-rolled steel ($\sigma_y=165\text{MPa}$), and it is loaded by the forces $F=0.55\text{kN}$, $P=8.0\text{kN}$ and $T=30\text{N.m}$.

Use the following stress concentrations at the wall
$K_{\text{axial}} = 1.8$
$K_{\text{bending}} = 1.6$
$K_{\text{torsion}} = 2.4$
$K_{\text{shear}} = 1.7$
No stress concentration Compute the factor of safety \( n \), based upon the distortion energy theory for the stress element \( A \).

\[
\sigma_x = \frac{Mc}{I} + \frac{P}{Area} = \frac{Fl\left(\frac{d}{2}\right)}{\pi d^4} + \frac{P}{\pi d^2} = \frac{32Fl}{\pi d^3} + \frac{4P}{\pi d^2}
\]

\[
\tau_{xy} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(30)}{\pi(0.020)^3} = 19.10\text{MPa}
\]

\[
\sigma_x = \frac{32(0.55)(10^3)(0.1)}{\pi(0.02)^3} + \frac{4(8)(10^3)}{\pi(0.02)^2} = 95.49\text{MPa}
\]

\[
\tau_{xy} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(30)}{\pi(0.020)^3} = 19.10\text{MPa}
\]

\[
\tau_{xy} = \frac{16T}{\pi d^3} + \frac{4V}{3A} = \frac{16(30)}{\pi(0.02)^3} + \frac{4(0.55)(10^3)}{3\left(\frac{\pi}{4}\right)(0.02)^2} = 21.43\text{MPa}
\]

\[
\tau_{Max} = \left[\left(\frac{25.47}{2}\right)^2 + (21.43)^2\right]^{1/2} = 27.55\text{MPa}
\]

\[
n = \frac{82.50}{27.55} = 2.99
\]

No stress concentration Compute the factor of safety \( n \), based upon the MSS theory for the stress element \( B \).

\[
\sigma_x = \frac{4P}{\pi d^2} = \frac{4(8)(10^3)}{\pi(0.02)^2} = 25.47\text{MPa}
\]

\[
\tau_{xy} = \frac{16T}{3A} = \frac{16(30)}{\pi(0.02)^3} + \frac{4(0.55)(10^3)}{3\left(\frac{\pi}{4}\right)(0.02)^2} = 21.43\text{MPa}
\]

\[
\tau_{Max} = \left[\left(\frac{25.47}{2}\right)^2 + (21.43)^2\right]^{1/2} = 27.55\text{MPa}
\]

\[
n = \frac{82.50}{27.55} = 2.99
\]
With stress concentration Compute the factor of safety \((n)\), based upon the distortion energy theory for the stress element \(A\).

\[
\sigma_x = K_{Bending} \frac{Mc}{I} + K_{Axial} \frac{P}{Area} = K_{Bending} \frac{F l \left( \frac{d}{2} \right)}{\pi d^4} + K_{Axial} \frac{P}{\pi d^2} = K_{Bending} \frac{32 F l}{\pi d^3} + K_{Axial} \frac{4P}{\pi d^2}
\]

\[
\sigma_x = 1.6 \times \frac{32(0.55)(10^3)(0.1)}{\pi (0.02)^3} + 1.8 \times \frac{4(8)(10^3)}{\pi (0.02)^2} = 112 \text{MPa} + 45.8 \text{MPa} = 157.8 \text{MPa}
\]

\[
\tau_{xy} = K_{Torsion} \times \frac{Tr}{J} = K_{Torsion} \times \frac{16T}{\pi d^3} = 2.4 \times \frac{16(30)}{\pi (0.020)^3} = 45.8 \text{MPa}
\]

\[
\sigma_{VM} = \sqrt{\left( \sigma_x^2 + 3\tau_{xy}^2 \right)} = \left[ 157.8^2 + 3(45.8)^2 \right]^{\frac{1}{2}} = 176.6 \text{MPa}
\]

\[
n = \frac{S_y}{\sigma_{VM}} = \frac{165}{176.6} = 0.93
\]
**With stress concentration** Compute the factor of safety \( (n) \), based upon the MSS theory for the stress element \( B \).

\[
\sigma_x = K_{axial} \times \frac{4P}{\pi d^2} = 1.8 \times \frac{4(8)(10^3)}{\pi(0.02)^2} = 45.85\text{MPa}
\]

\[
\tau_{xy} = K_{Torsion} \times \frac{16T}{\pi d^3} + K_{Shear} \times \frac{4V}{3A} = 2.4 \times \frac{16(30)}{\pi(0.02)^3} + 1.7 \times \frac{4(0.55)(10^3)}{3\left(\frac{\pi}{4}\right)(0.02)^2} = 45.83\text{MPa} + 3.97\text{MPa} = 49.8\text{MPa}
\]

\[
\tau_{Max} = \left[ \left( \frac{45.85}{2} \right)^2 + (49.8)^2 \right]^{1/2} = 54.82\text{MPa}
\]

\[n = \frac{82.5}{54.82} = 1.50\]
**Cantilever Case:** The discontinuity here is a simple circular hole, drilled through the depth of the beam on its centerline. The sketch shows the stress distribution at two sections of a cantilever beam, and illustrates the presence of stress concentration. At section A, the stress is uniform across the width of the beam, and calculable from the following relationship:

\[
\sigma_A = \frac{M_A \cdot c}{I_A} = \frac{6PL}{bt^2}
\]

where:
- \(\sigma\) = stress, psi (N/m²)
- \(M\) = bending moment, in-lbs (N-m)
- \(I\) = moment of inertia of beam cross section, in⁴ (m⁴)
- \(P\) = load, lbs (N)
- \(c\) = half-thickness of beam, in (m)
At section B, the nominal stress, based upon the net area of the section, is:

$$\sigma_B(\text{nom}) = \frac{M_B \cdot c}{I_B} = \frac{6Pl}{(b-d)t^2}$$

If the location of the hole is selected so that

$$\frac{l}{L} = \frac{b-d}{b}$$

the nominal stress at section B is the same as that at section A. The maximum stress at section B, however, is much greater, due to the stress concentration effect. As shown in the sketch, the maximum stress exists at the edge of the hole, on the transverse diameter, and the stress decreases rapidly with the distance from the hole. By definition the stress concentration factor, $K_t$, is the ratio of the maximum stress at the hole to the nominal stress at the same point. That is,

$$K_t = \frac{\sigma_B(\text{max})}{\sigma_B(\text{nom})} = \frac{6Pl}{(b-d)t^2}$$

Since the nominal stresses and the peak stress at the edge of the hole, are all uniaxial, the strain and stress are proportional. Thus, the stress concentration factor is equal to the ratio of the maximum to nominal strains at section B. Therefore,

$$K_t = \frac{\varepsilon_B(\text{max})}{\varepsilon_B(\text{nom})} = \frac{\varepsilon_B(\text{max})}{\varepsilon_A}$$
• Stress-strain behavior (Room T):

\[ TS_{\text{engineering}} \ll TS_{\text{perfect materials}} \]

**IDEAL VS REAL MATERIALS**

- perfect mat’l-no flaws
- carefully produced glass fiber
- typical ceramic
- typical strengthened metal
- typical polymer

\[ \sigma \]

\[ E/10 \]

\[ E/100 \]

\[ 0.1 \]

\[ \varepsilon \]
Theoretical Cohesive Strength of Metals

\[ \sigma = \sigma_{\text{Max}} \sin \left( \frac{2\pi x}{\lambda} \right) \]

\( \sigma_{\text{max}} \) is the theoretical cohesive strength

\( x = a - a_o \) is the displacement in atomic spacing in a lattice with a wavelength \( \lambda \). For small displacements \( \sin x = x \).
The ideal strength, $\sigma_i$, is given by:

$$\frac{E}{30} \leq \sigma_y \leq \frac{E}{4}$$
Hooke’s Law

High values of cohesive strength.

When fracture occurs in a brittle solid, all of the work expended in producing the fracture goes into the creation of two new surfaces.

\[ U_o = \int_0^{\lambda/2} \sigma_{max} \sin \left( \frac{2\pi x}{\lambda} \right) \delta x = \frac{\lambda \sigma_{Max}}{\pi} \]

But this energy is equal to the energy required to create the two new fracture surfaces.

Surface energy \( \gamma \) J/m\(^2\).
\[
\lambda = \frac{2\pi\gamma_s}{\sigma_{Max}} \quad \sigma_{Max} = \frac{\lambda}{2\pi} \frac{E}{a_o} \quad \sigma_{max} = \left(\frac{E\gamma_s}{a_0}\right)^{1/2}
\]

**Example:**
Determine the cohesive strength of a silica fiber, if \( E=95\text{GPa} \), \( \gamma_s=1000\text{erg/cm}^2 \) and \( a_0=1.6\text{ Angstroms} \).

\[
\gamma_s = 1000\text{erg.cm}^{-2} \times 10^{-3} = 1\text{J.m}^{-2}
\]
\[
a_o = 1.6 \times 10^{-10} \text{ m}
\]
\[
\sigma_{Max} = \left(\frac{E\gamma_s}{a_o}\right)^{1/2} = \left(\frac{95 \times 10^9 \times 1}{1.6 \times 10^{-10}}\right)^{1/2} = 24.4\text{GPa}
\]

Experience in high strength steels shows that fracture strength in excess of 2GPa is exceptional. Engineering materials typically have fracture stresses that are 10 to 100 times lower than the theoretical value.
Introduction to Fracture Mechanics

The central difficulty in designing against fracture in high-strength materials is that the presence of cracks can modify the local stresses to such an extent that the elastic stress analyses done so carefully by the designers are insufficient.

When a crack reaches a certain critical length, it can propagate catastrophically through the structure, even though the gross stress is much less than would normally cause yield or failure in a tensile specimen.

The term “fracture mechanics” refers to a vital specialization within solid mechanics in which the presence of a crack is assumed, and we wish to find quantitative relations between the crack length, the material’s inherent resistance to crack growth, and the stress at which the crack propagates at high speed to cause structural failure.
**Energy-Balance Approach (Griffith, 1921)**

Actual fracture strength in most materials are significantly lower than expected from bond strengths. **Flaw/cracks can amplify or concentrate stress!**

Max stress at the crack tip:

\[
\sigma_m = \sigma_0 \left[ 1 + 2 \left( \frac{a}{\rho_t} \right)^{1/2} \right]
\]

For long microcracks:

\[
\sigma_m = 2\sigma_0 \left( \frac{a}{\rho_t} \right)^{1/2}
\]

Stress concentration factor:

\[
K_t = \frac{\sigma_m}{\sigma_o} = 1 + 2 \sqrt{\frac{a}{\rho_t}}
\]

Developed by Inglis in 1913

---

**Figure 8.7** (a) The geometry of surface and internal cracks. (b) Schematic stress profile along the line \(X-X'\) in (a), demonstrating stress amplification at crack tip positions.

Large \(K_t\) promotes failure:

Surface crack are worse!

Minimize crack size \(a\) and maximize radius of curvature \(r_t\) if crack is unavoidable.
The Inglis solution poses a mathematical difficulty: in the limit of a perfectly sharp crack, the stresses approach infinity at the crack tip. This is obviously nonphysical (actually the material generally undergoes some local yielding to blunt the crack tip), and using such a result would predict that materials would have near zero strength: even for very small applied loads, the stresses near crack tips would become infinite, and the bonds there would rupture.

Griffith showed that the crack growth occurs when the energy release rate from applied loading is greater than the rate of energy for crack growth. Crack growth can be stable or unstable.
\[ \frac{\sigma_{\text{max}}}{S_a} = 1 + 2\frac{a}{b} \]
\[ \rho = \frac{b^2}{a} \]
\[ \sigma_{\text{max}} = S_a \left[ 1 + 2\sqrt{\frac{a}{\rho}} \right] \]
This approach assumes that the theoretical cohesive strength $\sigma_{\text{max}}$ can be reached locally at the tip of a crack while the average tensile strength is at much lower value.

$$\sigma_{\text{max}} = \sigma_{\text{aver}} \left(1 + 2 \sqrt{\frac{c}{\rho_t}} \right) \approx 2\sigma \sqrt{\frac{c}{\rho_t}}$$

$$\sigma_{\text{max}} = \left(\frac{E\gamma_s}{a_0}\right)^{1/2}$$

Where $\sigma$ is the nominal fracture stress.

$$\sigma \cong \sigma_{\text{max}} \sqrt{\frac{\rho_t}{4c}} \cong \left(\frac{E\gamma_s \rho_t}{a_0 \cdot 4c}\right)^{1/2}$$
The sharpest possible crack will be when $\rho = a_0$

\[
\sigma = \left( \frac{E\gamma_s}{4c} \right)^{\frac{1}{2}}
\]

The stresses near the crack tip are defined with the help of $K$.
Example

Calculate the nominal fracture stress for a brittle material with the following properties: \(E=100\,\text{GPa}\); \(\gamma_S=1\,\text{J/m}^2\); \(a_O=2.5\times10^{-10}\,\text{m}\) and a crack length of \(10^4a_O\)

\[
\sigma = \left( \frac{E\gamma_S}{4c} \right)^{\frac{1}{2}} = \left( \frac{100 \times 10^9 \times 1}{4 \times 2.5 \times 10^{-6}} \right)^{\frac{1}{2}} = 100\,\text{MPa} \approx \frac{E}{100}
\]

\(\gamma_S = 1\,\text{J/m}^2\) \quad \(a_O = 2.5 \times 10^{-10}\,\text{m}\)

\[
\sigma_{\text{Max}} = \left( \frac{E\gamma_S}{a_o} \right)^{\frac{1}{2}} = \left( \frac{100 \times 10^9 \times 1}{2.5 \times 10^{-10}} \right)^{\frac{1}{2}} = 20\,\text{GPa} \approx \frac{E}{5}
\]

Note a small crack produces a sharp decrease in the stress for fracture from \(E/5\) to \(E/100\)
Griffith used a result obtained by Inglis in 1913 that the change in strain energy due to an elliptical crack of size $a$ in an uniformly stressed plate is

$$\Delta U = \frac{\pi a^2 \sigma^2}{E}$$

and therefore the change in potential energy of the external load is twice as much.
**Griffith’s Theory**

Change of energy of a plate due to the introduction of a crack:

\[ U_{cracked} - U_{uncracked} = -\frac{2\pi a^2 \sigma^2}{E} + \frac{\pi a^2 \sigma^2}{E} + 4a\gamma \]

Minimizing the energy in relation to the crack length,

\[ \frac{\partial}{\partial a} \left( -\frac{\pi a^2 \sigma^2}{E} + 4a\gamma \right) = 0 \]

The critical stress (for plane stress conditions) is:

\[ \sigma = \sqrt{\frac{2E\gamma}{\pi a}} \]

\[ K = \sigma \sqrt{\pi a} \]

If the potential energy released as the crack grows is greater than the energy needed to create new crack surface, then the material will fracture.
**Griffith’s Theory**

Total energy of the system: \[ U = \text{Elastic Strain Energy} + \text{Energy Creating Crack} \]
\[ U = U_e + U_s \]

As the crack grows, \( U_e \) decreases and \( U_s \) increases.

For equilibrium

\[ \frac{\delta U}{\delta c} = 0 \]

\( \gamma \) = specific surface energy (energy per unit area required to break the bonds)

\[ \sigma_f = \sqrt{\frac{2E\gamma}{\pi c}} \]

Stress required to propagate a crack as a function of the size of the crack

\( \sigma_F \) = failure stress

\( E \) = Young’s modulus

\( c \) = crack half-length

---

Fig. 13. Griffith’s model for a crack propagating in a rod (a), and the energy partition for the process (b).
Importance of the Equation

Relates the size of the imperfection or defect to the tensile strength of the material.
Predicts that small imperfections are less damaging than large imperfections.
Strong materials have large $E$ and small $c$, strong is not the same as tough.
Tough materials imply a large energy absorption as crack advances, i.e. the energy required to produce new crack surface is high, that is high $\gamma$.

Can also be expressed in terms of $G_C = \text{elastic energy release rate, or crack driving force}$, (where the use of the word rate means per increment of crack length not time).
$G_C$ has dimensions of energy/unit plate thickness/unit crack extension.

$$\sigma_f = \sqrt{\frac{2E\gamma}{\pi c}}$$ Critical stress for plane stress conditions
Two crack tips \(2G_C = \frac{dU}{dc} = \frac{dW}{dc} = 4\gamma\)

Critical stress for plane stress conditions:

\[
\sigma_C = \sqrt{\frac{E G_C}{\pi c}}
\]

A closer look to the definition of a Crack

According to the Griffith criterion:

\[
\sigma_f = \sqrt{\frac{2E\gamma}{\pi c}}
\]

According to the Cohesive Strength:

\[
\sigma_f \approx \sigma_{\text{max}} \sqrt{\frac{\rho_t}{4c}} \approx \left(\frac{E\gamma_s \rho_t}{a_0 4c}\right)^{1/2}
\]

Equating both expressions:

\[
\frac{\rho_t}{a_0} = \frac{8}{\pi} \approx 2.5
\]

When the curvature of the crack is lower than \(3a_0\) then Griffiths should be used.
Fracture toughness is an indication of the amount of stress required to propagate a preexisting flaw. It is a very important material property since the occurrence of flaws is not completely avoidable in the processing, fabrication, or service of a material/component.

![Diagram of crack propagation modes](image)

Fig. 1.5. The three crack propagation modes.

A parameter called the **stress-intensity factor** \( (K) \) is used to determine the fracture toughness of most materials. A Roman numeral subscript indicates the mode of fracture.
The stress distribution at the crack tip in a thin plate for an elastic solid in terms of the coordinates is given by:

\[
\begin{align*}
\sigma_x &= \sigma \left( \frac{a}{2r} \right)^{1/2} \left[ \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right] \\
\sigma_y &= \sigma \left( \frac{a}{2r} \right)^{1/2} \left[ \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right] \\
\tau_{xy} &= \sigma \left( \frac{a}{2r} \right)^{1/2} \left[ \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right]
\end{align*}
\]

For an orientation directly ahead of the crack \(\Theta=0\)

\[
\sigma_x = \sigma_y = \sigma \left( \frac{a}{2r} \right)^{1/2}
\]

For an infinite wide plate the relationship is:

\[\tau_{xy} = 0\]

\[
K = \sigma \sqrt{\pi a}
\]

Stress Intensity Factor
The stress intensity factor, $K$, is the enhancement at the crack tip of the tensile stress applied normal to the crack, for a sharp flaw in an infinite plate. The stress distribution is usually expressed in terms of this stress intensity factor, $K$.

\[ K = \sigma \sqrt{\pi a} \]

\[
\sigma_x = \frac{K}{\sqrt{2\pi r}} \left[ \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right]
\]

\[
\sigma_y = \frac{K}{\sqrt{2\pi r}} \left[ \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right]
\]

\[
\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \left( \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)
\]

**Stress Intensity Factors - Modes**

- $\rho_t$ at a crack tip is very small!
• Result: crack tip stress is very large.

• Crack propagates when: the tip stress is large enough to make:

\[ K \geq K_c \]
A properly determined value of $K_{lc}$ represents the fracture toughness of the material independent of crack length, geometry or loading system.

$K_{lc}$ is a material property

Specimens of a given ductile material, having standard proportions but different absolute size (characterized by thickness) give rise to different measured fracture toughness. Fracture toughness is constant for thicknesses exceeding some critical dimension, $b_o$, and is referred to as the plane strain fracture toughness, $K_{lc}$.
$K_{IC}$: It is a true material property, independent of size. As with materials' other mechanical properties, fracture toughness is tabulated in the literature, though not so extensively as is yield strength for example.
Plane-Strain Fracture Toughness Testing
When performing a fracture toughness test, the most common test specimen configurations are the single edge notch bend (SENB or three-point bend), and the compact tension (CT) specimens. It is clear that an accurate determination of the plane-strain fracture toughness requires a specimen whose thickness exceeds some critical thickness \( B \). Testing has shown that plane-strain conditions generally prevail when:

\[
B \geq 2.5 \left( \frac{K_{IC}}{\sigma_y} \right)^2
\]
**GEOMETRY, LOAD, & MATERIAL**

- **Condition for crack propagation:**
  \[ K \geq K_c \]

  **Stress Intensity Factor:**
  --Depends on load & geometry.

  **Fracture Toughness or Critical SIF:**
  Material parameter, Depends on the material, temperature, environment, & rate of loading.

- **Values of \( K \) for some standard loads & geometries:**

  **units of \( K \):**
  MPa\(\sqrt{m}\)
  or ksi\(\sqrt{in}\)

  **Adapted from Fig. 8.8,**
  *Callister 6e.*

  \[ K = \sigma \sqrt{\pi a} \]

  \[ K = 1.1 \sigma \sqrt{\pi a} \]
Uses of Plane-Strain Fracture Toughness

$K_{IC}$ values are used to:
(a) determine the critical crack length when a given stress is applied to a component.
(b) to calculate the critical stress value when a crack of a given length is found in a component.

Design Against Crack Growth

- Crack growth condition:

$$a_c = \frac{1}{\pi} \left( \frac{K_{IC}}{\sigma Y} \right)^2$$

$$\sigma_c \leq \frac{K_{IC}}{Y \sqrt{\pi a}}$$

$K \geq K_c$

- Largest, most stressed cracks grow first!
**Result 1:** Max flaw size dictates design stress.

\[ \sigma_{\text{design}} < \frac{K_c}{\gamma \sqrt{\pi a_{\text{max}}}} \]

**Result 2:** Design stress dictates max. flaw size.

\[ a_{\text{max}} < \frac{1}{\pi} \left( \frac{K_c}{\gamma \sigma_{\text{design}}} \right)^2 \]
Crack Geometry (Y or F)

\[ K = \frac{S_g \sqrt{\pi a}}{(a/b \leq 0.4)} \]

\[ F = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1 - \alpha}} \quad (h/b \geq 1.5) \]

\[ K = 1.12S_g \sqrt{\pi a} \quad (a/b \leq 0.6) \]

\[ F = \left(1 + 0.122\cos^2\frac{\pi \alpha}{2}\right) \sqrt{\frac{2}{\pi \alpha}} \tan \frac{\pi \alpha}{2} \quad (h/b \geq 2) \]

\[ K = 1.12S_g \sqrt{\pi a} \quad (a/b \leq 0.13) \]

\[ F = 0.265(1-\alpha)^{1.5} + \frac{0.857 + 0.265\alpha}{(1-\alpha)^{1.3}} \quad (h/b \geq 1) \]
Normalized Crack Size \( \alpha = \frac{a}{w} \)

<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th>Formula 1</th>
<th>Formula 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Long strip, central crack, tensile stress</td>
<td>( K_I = \sigma Y \sqrt{\pi a} )</td>
<td>( Y = \sqrt{\sec(\pi \alpha/2)} )</td>
</tr>
<tr>
<td>(b)</td>
<td>Long strip, edge crack, tensile stress</td>
<td>( K_I = \sigma Y \sqrt{\pi a} )</td>
<td>( Y = \frac{1.12 + \alpha (2.91 \alpha - 0.64)}{1 - 0.93 \alpha} )</td>
</tr>
<tr>
<td>(c)</td>
<td>Finite plate, edge crack, tensile force (CTS)</td>
<td>( K_I = \frac{P}{bw} Y \sqrt{\pi a} )</td>
<td>( Y = \frac{5.23 + \alpha (5.16 \alpha - 5.88)}{1 - 1.07 \alpha} )</td>
</tr>
<tr>
<td>(d)</td>
<td>Long strip, edge crack, pure bending</td>
<td>( K_I = \left( \frac{6M}{bw^2} \right) Y \sqrt{\pi a} )</td>
<td>( Y = \frac{1.12 + \alpha (2.62 \alpha - 1.59)}{1 - 0.7 \alpha} )</td>
</tr>
<tr>
<td>(e)</td>
<td>Finite plate, edge crack, bending (SEN)</td>
<td>( K_I = \frac{6P}{bw} Y \sqrt{\pi a} )</td>
<td>( Y = \frac{1.12 + \alpha (3.43 \alpha - 1.89)}{1 - 0.55 \alpha} )</td>
</tr>
<tr>
<td>(f)</td>
<td>Long rod, circumferential crack, tensile stress</td>
<td>( K_I = \sigma Y \sqrt{\pi a} )</td>
<td>( Y = \frac{1.12 + \alpha (1.30 \alpha - 0.88)}{1 - 0.92 \alpha} )</td>
</tr>
</tbody>
</table>
Example 1:
A ceramic has a strength of 300MPa and a fracture toughness of 3.6MPa.m^{0.5}. What is the largest-size internal crack that this material can support without fracturing?

\[ K_c = \sigma \sqrt{\pi a} \]

\[ a = \frac{K_c^2}{\pi \sigma^2} = \frac{(3.6MPa.m^{0.5})^2}{\pi \cdot (300MPa)^2} = 4.58 \times 10^{-5} m \]
**Example 2.**

A large sheet containing a 50 mm long crack fractures when loaded to 500 MPa. Determine the fracture load of a similar sheet with a 100 mm crack.  

[ 354 MPa ]

**Solution**

Large sheet, so there are no geometry effects \( Y=1 \)

\( K \rightarrow K_{lc} \) when \( \sigma=500\text{MPa}, a=25\text{mm} \)

\[
K_{lc} = \sigma \sqrt{\pi a} = 500 \sqrt{\pi \times 0.025} = 140\text{MPa} \cdot \sqrt{m} \\
\sigma_c = \frac{140\text{MPa} \cdot \sqrt{m}}{\sqrt{\pi \times 0.050}} = 354\text{MPa}
\]
Example 3
Rocket motor casings may be fabricated from either of two steels:
(a) low alloy steel yield 1.2 GPa toughness 70 MPa√m,
(b) maraging steel yield 1.8 GPa toughness 50 MPa√m
The relevant Code specifies a design stress of yield/1.5. Calculate the minimum defect size which will lead to brittle fracture in service for each material, and comment on the result (this last is important).
Assume that the configuration is essentially a large plate with $Y=1$.
[ 4.9, 1.1 mm ]

Solution
A safety factor of 1.4 implies that the casing will be put into service with a designed stress of
\[ \sigma = \frac{S_y}{n} \]
Criticality will occur when $K \geq K_{lc}$
\[ K_{Ic} = \sigma \sqrt{\pi a_c} \]

\[ a_c = \frac{1}{\pi} \left( \frac{K_{Ic}}{\sigma} \right)^2 = \frac{1}{\pi} \left( \frac{nK_{Ic}}{S_Y} \right)^2 \]

Evaluating this for the two materials

\[ 2a_c = \frac{2}{\pi} \left( \frac{1.5 \times 70}{1200} \right)^2 \times 10^3 = 4.9 \text{mm} \]

\[ 2a_c = \frac{2}{\pi} \left( \frac{1.5 \times 50}{1800} \right)^2 \times 10^3 = 1.1 \text{mm} \]

Since the design stress is a constant fraction of the yield, the material toughness must increase proportional to yields. If the materials are to possess the same defect tolerance, an inverse relationship occurs, that is, a high strength materials does not necessarily has a good defect tolerance when brittle fracture is possible.
As an example consider an eccentric load $P$ applied at a distance $e$ from the centerline of a component with an edge crack. This eccentric load is statically equivalent to the combination of a centrally applied load $P$ and a bending moment $M=Pe$. Then using the stress intensity factors for (1) an edge crack in a finite width strip in tension and (2) an edge crack in a finite width strip subjected to pure bending, the stress intensity factor for the eccentrically loaded strip with an edge crack can be easily calculated.
Example 4.

The bar of 100 x 20 mm rectangular cross-section is loaded by a force of 250 kN as shown. Determine the critical crack length (a) if the toughness is 50 MPa√m. [14 mm]

Solution

The crack size must be found by trial and error. Let

\[ \alpha = \frac{a}{w} = \frac{a}{100\text{mm}} = \frac{a}{0.1\text{m}} = 10a \]

Tension (Case b):

\[ \sigma = \frac{P}{A} = \frac{250000\text{N}}{0.1 \times 0.02} = 125\text{MPa} \]

\[ K_{Ic} = \sigma \sqrt{\pi a Y_t} = 125\text{MPa} \left( \sqrt{\pi \times 0.1\alpha} \right) Y_t \]
\[ Y_t = \frac{1.12 + \alpha(2.91\alpha - 0.64)}{1-0.93\alpha} \]

Bending (Case d):

\[ \sigma = \frac{6M}{bw^2} = \frac{6 \times 250000 \times 10}{20 \times 100^2} = 75\text{MPa} \]

\[ K_{Ib} = \sigma\left(\sqrt{\pi \times 0.1\alpha}\right)Y_b \]

\[ Y_b = \frac{1.12 + \alpha(2.62\alpha - 1.59)}{1-0.7\alpha} \]

<table>
<thead>
<tr>
<th>Trial ( \alpha )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.15</th>
<th>0.14</th>
<th>0.143</th>
<th>( \alpha_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{It} )</td>
<td>26.5</td>
<td>42.7</td>
<td>34.4</td>
<td>32.8</td>
<td>33.2</td>
<td></td>
</tr>
<tr>
<td>( K_{Ib} )</td>
<td>14.1</td>
<td>19.8</td>
<td>17.1</td>
<td>16.5</td>
<td>16.7</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>40.6</td>
<td>62.5</td>
<td>51.5</td>
<td>49.3</td>
<td>50.0</td>
<td>( K_{Ic} )</td>
</tr>
</tbody>
</table>

\textit{The critical crack length is 14.3mm}
Example 6:

The stress intensity for a partial through thickness flaw is given by the expression:

$$K = \sigma \sqrt{\pi a} \sqrt{\sec \frac{\pi a}{2t}}$$

Where $a$ is the depth of penetration of the flaw through a wall thickness $t$. If the flaw is 5mm deep in a wall of 0.5in thick, determine whether the wall will support a stress of 25000psi if it is made of 7075-T6 aluminum alloy.
**Solution:**

From table 11.1
Yield Strength = 500MPa
Fracture Toughness = 24MPa√m

Determine the critical stress level to make the 5mm flaw propagate to failure in this material.

\[
\sec \frac{\pi a}{2t} = \sec \left( \frac{\pi \times 5 \times 10^{-3}}{2.54 \times 10^{-2}} \right) = 1.227
\]

\[
\sigma = \frac{K_{lc}}{\sqrt{\pi a}} \left( \frac{1}{1.227} \right)^{1/2} = \frac{24}{\sqrt{\pi \times 5 \times 10^{-3}} \sqrt{1.227}} = 172.6MPa
\]

But the applied stress is 25000psi = 172.4MPa, Therefore the flaw will propagate as a brittle fracture.
Example 7: Design of a Pressure Vessel

Two design considerations to avoid catastrophic failure:

a. Yield before break.
b. Leak before break

a. Yield before break

Hoop stress:

\[ \sigma = \frac{PR}{t} \]

For yield to occur

\[ \sigma_y = \frac{PR}{t} \]
For fracture to occur:

\[ \sigma = \frac{K_c}{\sqrt{\pi a_c}} \]

Critical size just before propagation:

\[ a_c = \frac{1}{\pi} \left( \frac{K_c}{\sigma_y} \right)^2 \]

Choose a material with high

\[ \left( \frac{K_c}{\sigma_y} \right)^2 \]
**b. Leak before fracture**

The crack must be stable (do not grow) when the crack size equals the wall thickness.

It must also contain the pressure without yielding.

Hence:

\[
P = \frac{K_c \sqrt{t}}{R \sqrt{\pi}} \quad \Rightarrow \quad P^2 = \frac{K_c^2 t}{R^2 \pi} \quad \Rightarrow \quad P = \frac{1}{\pi R} \left( \frac{K_c^2}{\sigma_y} \right)
\]

Choose a material with highest

\[
\left( \frac{K_c}{\sigma_y} \right)
\]
**Example 8:**

A thin-wall pressure vessel is made of Ti-6Al-4V alloy. The internal pressure produces a circumferential hoop stress of 360MPa. The crack is a semielliptical surface crack oriented with the major plane of the crack perpendicular to the uniform tensile hoop stress. Find the size of the critical crack which will cause rupture of the pressure vessel with a wall thickness of 12mm. For this type of loading and geometry the stress intensity factor is given by:

\[
K_I^2 = \frac{1.21a\pi\sigma^2}{Q}
\]

where:
- \(a\) = surface crack depth
- \(\sigma\) = applied stress normal
- \(Q\) = Flaw shape parameter

\[
Q = \phi^2 - 0.212\left(\frac{\sigma}{\sigma_0}\right)^2
\]

where \(\phi^2\) = A complex flow shape parameter.
Table 11-1  Typical values of $K_{1c}$

<table>
<thead>
<tr>
<th>Material</th>
<th>Yield strength, MPa</th>
<th>Fracture toughness $K_{1c}$, MPa/(\text{m})</th>
</tr>
</thead>
<tbody>
<tr>
<td>4340 steel</td>
<td>1470</td>
<td>46</td>
</tr>
<tr>
<td>Maraging steel</td>
<td>1730</td>
<td>90</td>
</tr>
<tr>
<td>Ti-6Al-4V</td>
<td>900</td>
<td>57</td>
</tr>
<tr>
<td>2024-T3 Al alloy</td>
<td>385</td>
<td>26</td>
</tr>
<tr>
<td>7075-T6 Al alloy</td>
<td>500</td>
<td>24</td>
</tr>
</tbody>
</table>

\[
\frac{\sigma}{\sigma_0} = \frac{360}{900} = 0.4
\]

Taken from Dieter

\[
Q = \phi^2 - 0.212 \left(\frac{\sigma}{\sigma_0}\right)^2
\]

where $Q = \phi^2$ is a complex flaw shape parameter.
Solution:

Assume \(2c=2a\) and 

Then \(Q=2.35\) and

\[
\frac{\sigma}{\sigma_o} = 0.4
\]

\[
a_c = \frac{K_I^2Q}{1.21\pi\sigma^2} = \frac{57^2 \times 2.35}{1.21\pi(360)^2} = 0.01549m = 15.5mm
\]

Note that the critical crack depth, 15.5mm is greater than the thickness of the vessel wall, 12mm. The crack will break through the vessel wall and the fluid will leak ("leak-before-break" condition). However, if the crack is very elongated \(a/2c = 0.05\) then \(Q=1.0\) and the \(a_c=6.6mm\). In this case the vessel will fracture when the crack had propagated about half-way through the wall thickness.
Example 9:

You are involved in the design and manufacture of 6.6m diameter (D) rocket motor cases with a wall thickness (t) of 18.5 mm, and an operational pressure (P) of 6.6 MPa. These components are presently manufactured from a Grade 200 maraging steel with a yield strength of 1515MPa and $K_{1C} = 136.5\text{MPa.m}^{\frac{1}{2}}$. In order to save weight, a design engineer has proposed changing to a Grade 250 maraging steel with a yield strength of 1650MPa and a plane strain fracture toughness value of 72.5MPa.m$^{\frac{1}{2}}$, and has requested an fracture analysis of allowable defect size against failure stress. Failure of the motor case can be assumed to occur from embedded elliptical defects orientated perpendicular to the hoop stress. Typical elliptical defects resulting from the welding process can be detected by NDT, and are known to occur with sizes up to 5.5mm by 35.5mm. Determine design data for these alloys of fracture stress against allowable defect size, over a range of major axis lengths from 20mm to 50mm.
To provide the fracture analysis requested by the designer, all we need do is calculate a table of
fracture stress against defect size for the two materials. Although the calculations are based on
the length of the semi-minor axis \((a)\), it is useful to show the full length of the minor axis \((2a)\)
in the table, as this would be the parameter obtained from NDT. The fracture stress
 corresponding to likely weld defects can then be compared with the design (hoop) stress and the
appropriate recommendation made.
The hoop stress in this component is:
\[
\sigma_H = \frac{pD}{2t} = \frac{6.6 \times 6.6}{2 \times 0.0185} = 1177 \text{ MPa}
\]

As the defect is embedded, no additional contribution to stress intensity arises from the internal pressure in the motor case (see Theory Card in this question for information on superposition of \(K\) values). Thus the design stress is synonymous with the hoop stress. The
table below gives fracture stress for various values of \(2a\) corresponding to the range of defect sizes of interest \((20 \text{ mm} < 2c < 50 \text{ mm})\).

<table>
<thead>
<tr>
<th>Alloy</th>
<th>Defect Size (2a) mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade 200</td>
<td>3.4 4.5 5.0 5.5 6.0 6.5 7.0 7.5 12.1</td>
</tr>
<tr>
<td>Grade 250</td>
<td>1177 1026 973 928 888 854 822 795</td>
</tr>
</tbody>
</table>
This column in bold indicates the fracture stresses corresponding to the presence of typical weld defects. The Grade 200 steel can tolerate cracks of up $2a = 12.1$ mm at the design stress, while the fracture stress for a crack with $2a = 5.5$ mm is greater than the yield strength. (Note, however, that we do not know what temperature the given material property data refer to, or what the operating temperatures of the motor case are - this is important information, as fracture toughness and yield strength are functions of temperature). The Grade 250 steel, however, can only tolerate cracks with $2a = 3.4$ mm and would suffer fracture at the design stress. The recommendation to the design engineer would be to continue with the use of the Grade 200 steel.
**Example 10:**

Power generation pressure vessels are usually thick walled, operate at a range of temperatures from ambient to elevated and are required to be fail-safe (leak-before-break). In one particular case, a spherical pressure vessel is proposed which will operate at an internal pressure \( p \) of 40 MPa and at temperatures from 0°C to 300°C. The proposed wall thickness \( t \) is 100 mm and the diameter \( D \) is 2 m. Two candidate steel alloys have been suggested:

**Steel A** steel: For this steel, \( K_C = (150 + 0.05T) \) MPa m\(^{1/2} \) where \( T \) is operating temperature in degrees centigrade, and the yield strength varies in a linear fashion from 549 MPa at 0°C to 300 MPa at 300°C.

**Steel B** steel: Here \( K_C = (100 + 0.25T) \) MPa m\(^{1/2} \), and the yield strength varies linearly from 650 MPa at 0°C to 500 MPa at 300°C.

Graphically determine, by inspection, the range of temperatures over which each of these alloys would have the highest safety factor with respect to fast fracture.

Through-thickness cracks can be assumed to be critical and the stress intensity factor for such cracks in this geometry is given by:

\[
K_C^2 = \frac{\pi \sigma^2 a}{1 - 0.5 \left( \frac{\sigma}{\sigma_{YS}} \right)^2}
\]

The membrane stress in the pressure vessel wall may be taken as \( pD/4t \).
Solving this question is simply a matter of calculating the required values of fracture toughness $K_C$ to avoid fracture at various temperatures in the operating range, say 0°C, 100°C, 200°C and 300°C for both steels.

The membrane stress is simply found from: $$\sigma = \frac{pD}{4t} = \frac{40 \times 2.0}{4 \times 0.1} = 200 \text{ MPa}$$

Although the design case is based on leak-before-break, the amount of pressure relief caused by a through-thickness crack is unknown. It is therefore conservative to assume that the internal pressure will load the crack surfaces, hence the total stress intensity factor will be calculated using the sum of the membrane stress and the internal pressure, i.e. 240 MPa.
Leak-before-break design requires the pressure vessel to tolerate a through-thickness crack of total length \(2a\) = the surface length \((2c)\) of the pre-cursor semi-elliptic crack. As we have no information regarding crack ellipticity, we will have to assume that it was semi-circular and hence \(2a = 2t\), where \(t\) is the wall thickness. Therefore the required values of toughness are found from:

\[
K_C^2 = \frac{\pi \times 240^2 \times 0.1}{1 - 0.5 \left( \frac{240}{\sigma_{YS}} \right)^2}
\]
The table below gives required and available toughness values for the two alloys.

<table>
<thead>
<tr>
<th>Steel</th>
<th>Yield Strength MPa</th>
<th>0°C</th>
<th>100°C</th>
<th>200°C</th>
<th>300°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>540</td>
<td>460</td>
<td>380</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>Required $K_C$ MPa m$^{\frac{1}{2}}$</td>
<td>141.7</td>
<td>144.7</td>
<td>150.3</td>
<td>163.1</td>
</tr>
<tr>
<td></td>
<td>Actual $K_C$ MPa m$^{\frac{1}{2}}$</td>
<td>150</td>
<td>155</td>
<td>160</td>
<td>165</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>650</td>
<td>600</td>
<td>550</td>
<td>500</td>
</tr>
<tr>
<td></td>
<td>Required $K_C$ MPa m$^{\frac{1}{2}}$</td>
<td>139.4</td>
<td>140.2</td>
<td>141.4</td>
<td>143.0</td>
</tr>
<tr>
<td></td>
<td>Actual $K_C$ MPa m$^{\frac{1}{2}}$</td>
<td>100</td>
<td>125</td>
<td>150</td>
<td>175</td>
</tr>
</tbody>
</table>
This data is plotted in the graph below. By inspection, the toughness values for Steel A are highest and therefore the safety margin greatest, up to about 212ºC. Above that temperature, Steel B has become the best choice because of its very steep increase in toughness with temperature.
**Example 5:**

Welded plates, 10 mm thick, are subjected to bending (see figure). Crude manufacture leads to the expectation of 2 mm cracks extending right along the weld root. Multiple service failures occur when the deposition properties are as (b) below. Would a change to (a) or to (c) alleviate the problem?

<table>
<thead>
<tr>
<th>deposition</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>yield ( MPa )</td>
<td>600</td>
<td>800</td>
<td>1000</td>
</tr>
<tr>
<td>toughness ( MPa√m )</td>
<td>120</td>
<td>90</td>
<td>60</td>
</tr>
</tbody>
</table>
Solution

Elastic

Case \(_{(d)}\)

\[
Y = \frac{1.12 + \alpha(2.62\alpha - 1.59)}{1 - 0.7\alpha} \quad \alpha = \frac{a}{w} = \frac{2}{10} = 0.2 \quad Y = 1.054
\]

\[
K_{lc} = \frac{6M}{bw^2} Y \sqrt{\pi a} \Rightarrow \frac{M}{bw^2} = 1.994K_{lc}
\]

Plastic

\[
z_1 + z_2 = w - a = (1 - \alpha)w
\]

\[
\Sigma M_{LH,\text{Center}} = M + b z_1 S_Y \left( a + \frac{z_1}{2} - \frac{w}{2} \right) - b z_2 S_Y \left( w - \frac{z_2}{2} - \frac{w}{2} \right) = 0
\]

\[
\frac{4M}{bw^2 S_Y} = (1 - \alpha)^2 \quad \frac{M}{bw^2} = (1 - \alpha)^2 \frac{S_Y}{4}
\]
<table>
<thead>
<tr>
<th>Material</th>
<th>(b)</th>
<th>(a)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic M/bw²</td>
<td>90x1.994</td>
<td>120x1.994</td>
<td>60x1.994</td>
</tr>
<tr>
<td>Plastic M/bw²</td>
<td>128</td>
<td>96</td>
<td>160</td>
</tr>
</tbody>
</table>