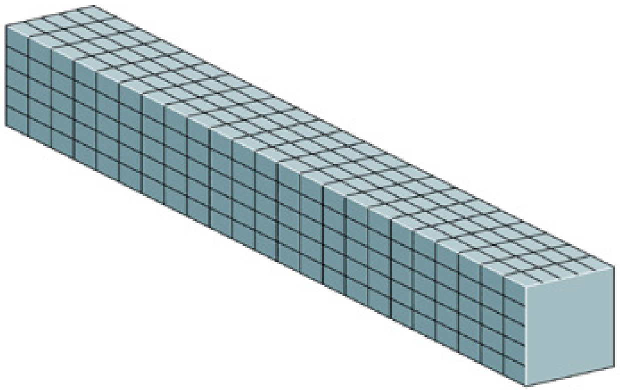
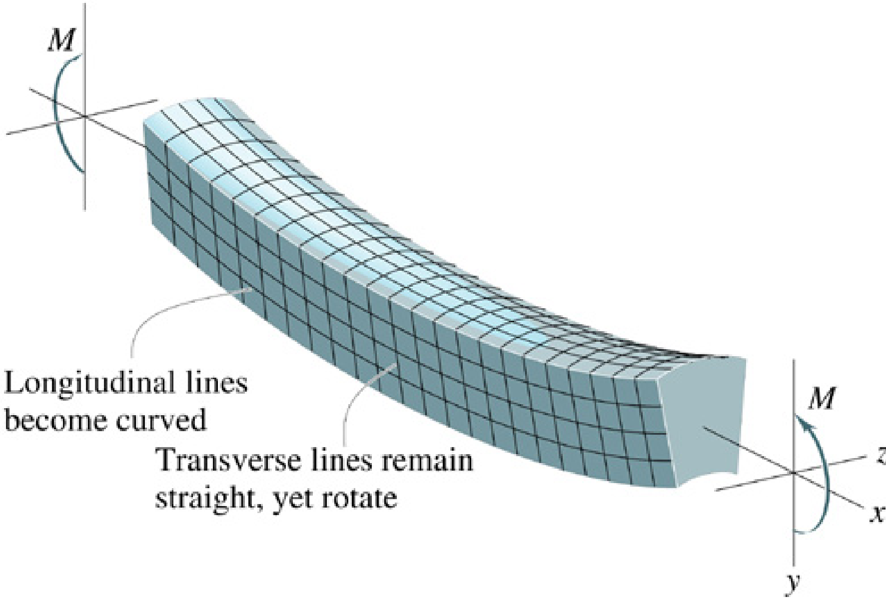
Bending Stress and Strain

DEFLECTIONS OF BEAMS

When a beam with a straight longitudinal axis is loaded by lateral forces, the axis is deformed into a curve, called the ***deflection curve*** of the beam.


(a)

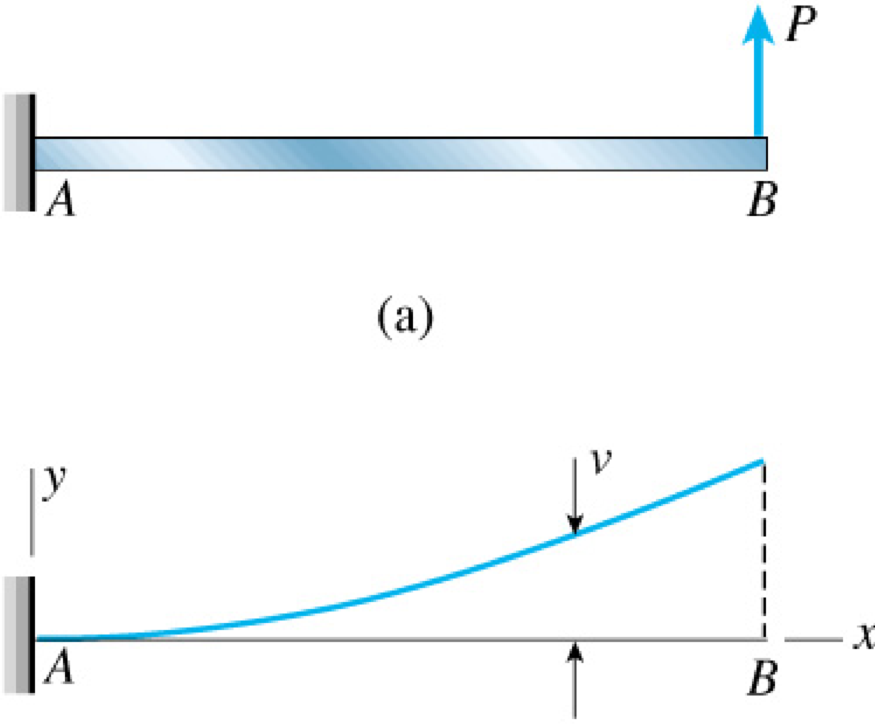
We will determine the equations fo finding the deflection curve and also find the deflections at specific points along the axis of the beam.



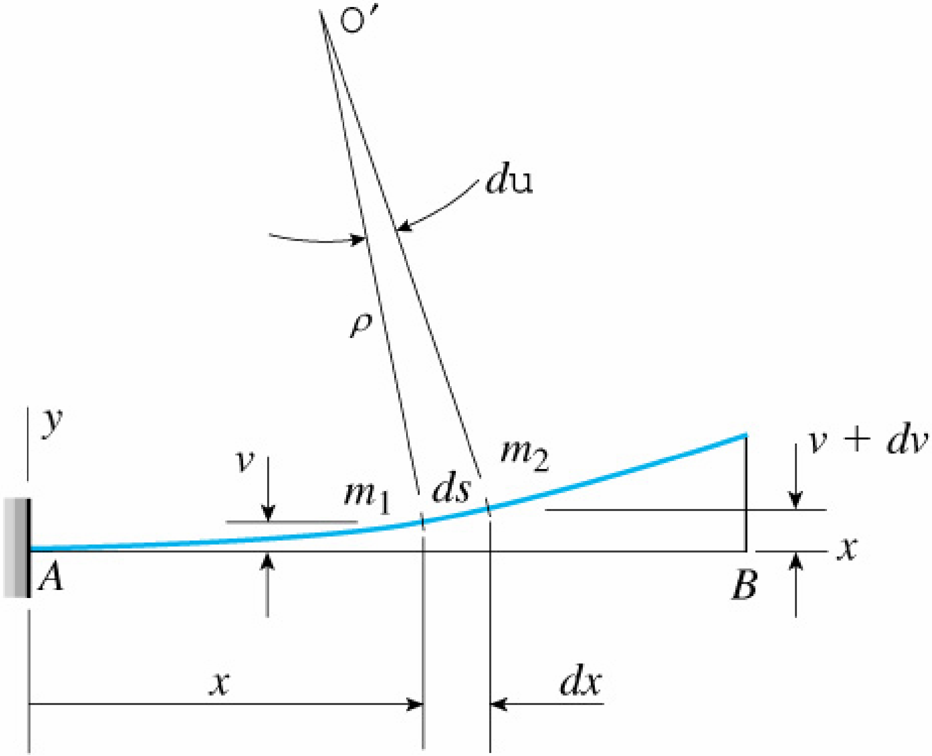
(b)

Differential equations of the deflection curve

Consider a cantilever beam with a concentrated load acting upward at the free end. Under the action of the load, the axis of the beam deforms into a curve. The reference axes have their origin at the fixed end of the beam. ***X*** is positive to the right and ***y*** is positive upwards.

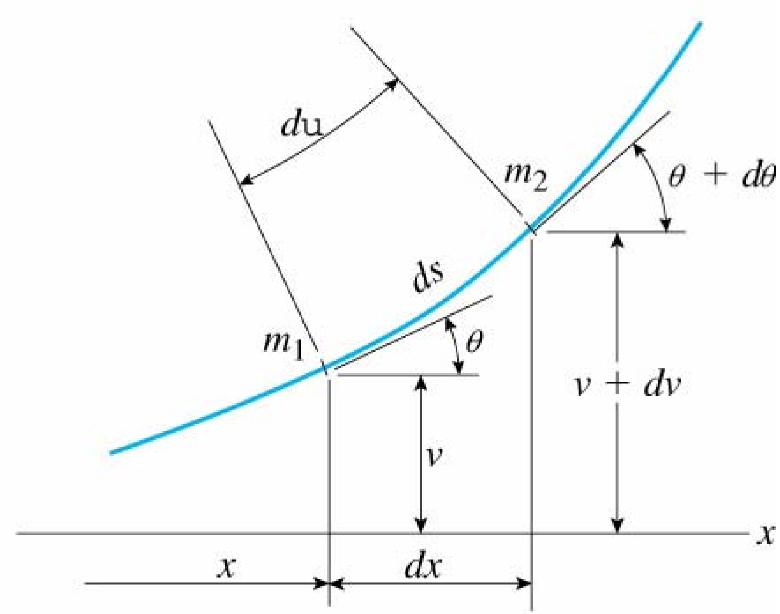


The ***deflection v*** is the displacement in the ***y*** direction of any point on the axis of the beam. We must express ***v*** as a function of the coordinate ***x***.

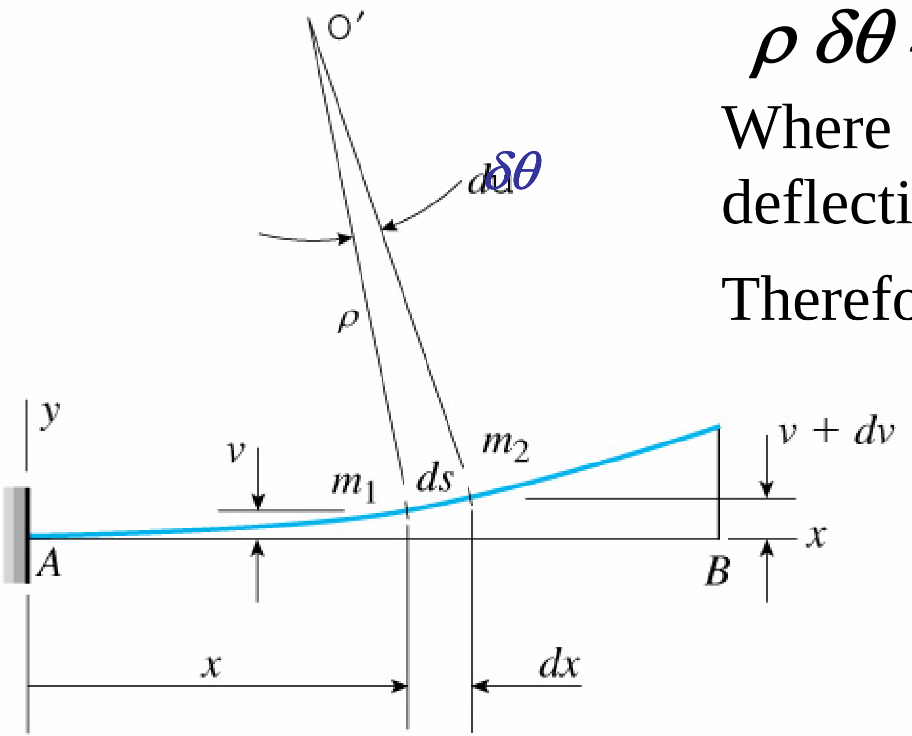


Consider the points m_1 and m_2 located at a distance x and $x + \delta x$ from the origin respectively. Point m_1 has a deflection equal to v and point m_2 has a deflection equal to $v + \delta v$, where δv is the increment in deflection as we move from m_1 to m_2.

When the beam is bent, there is not only a deflection at each point along the axis but also a rotation. The ***angle of rotation*** θ ***(also known as angle of inclination and angle of slope)*** of the axis of the beam is the angle between the ***x-axis*** and the tangent to the deflection curve.



The angle of rotation is θ for point m_1 and $\theta + \delta\theta$ for point m_2 , i.e., the angle between the lines normal to the tangents at points m_1 and m_2 is $\delta\theta$. The point of intersection of these normals is the ***center of curvature O'*** and the distance ***O'*** to m_1 is the ***radius of curvature*** ρ.



$$\rho\,\delta\theta = \delta\sigma$$

Where δs is the distance along the deflection curve between m_1 and m_2.

Therefore the curvature

$$\kappa = \frac{1}{\rho} = \frac{\delta\theta}{\delta s}$$

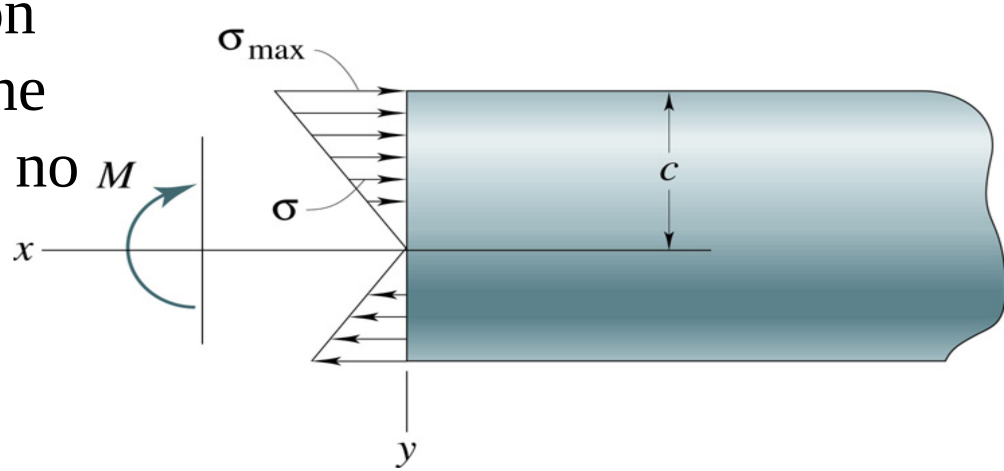
The slope of the deflection curve is the first derivative $\delta v / \delta x$ and is equal to $\tan\ \theta$.

Similarly, $\sin\theta = \delta v / \delta s$ and $\cos\theta = \delta x / \delta s$

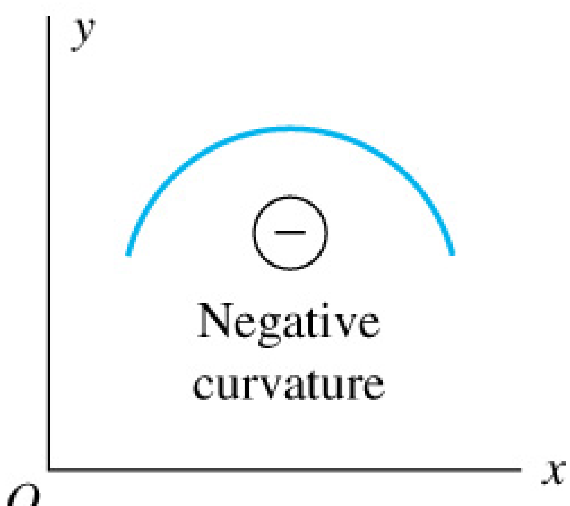
If the angle of rotation θ is very small then $\delta s \sim \delta x$; the curvature $k = 1/\rho = \delta\theta/\delta x$; and $\tan\theta \sim \theta = dv/dx$.

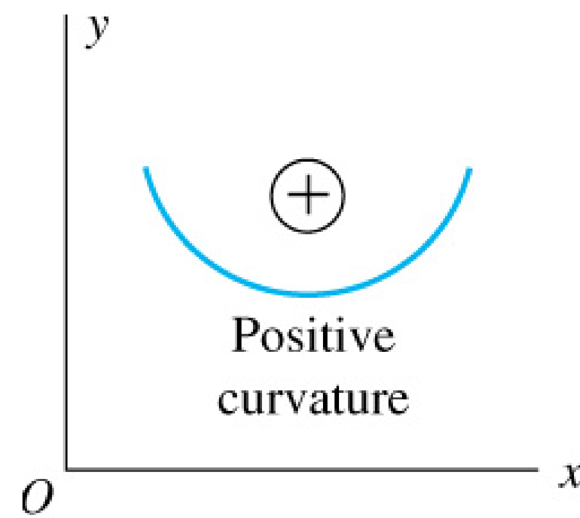
Thus, if the rotations of the beam are very small, we can assume that the angle of rotation θ (in radians) and the slope $\delta v/\delta x$ are equal.

The bending moment causes tension at the bottom and compression at the top. At the ***neutral surface*** there is no tension or compression, then



$$\varepsilon = -\frac{y}{c}\varepsilon_{Max} \qquad \sigma = -\frac{y}{c}\sigma_{Max}$$





Taking the derivative of θ with respect to x:

$$\frac{\delta\theta}{\delta x} = \frac{\delta^2 \upsilon}{\delta x^2}$$

If the material is linearly elastic and it obeys Hooke's law, the curvature is:

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} = \frac{\delta^2 \upsilon}{\delta x^2}$$

M is the bending moment and EI is the flexural rigidity of the beam. This equation is known as the differential equation of the deflection curve. It can be integrated in each particular case to find the deflection v, provided the bending moment M and flexural rigidity EI are known as functions of x.

$$\frac{\delta^2 \upsilon}{\delta x^2} = \frac{M}{EI}$$

Sign Conventions

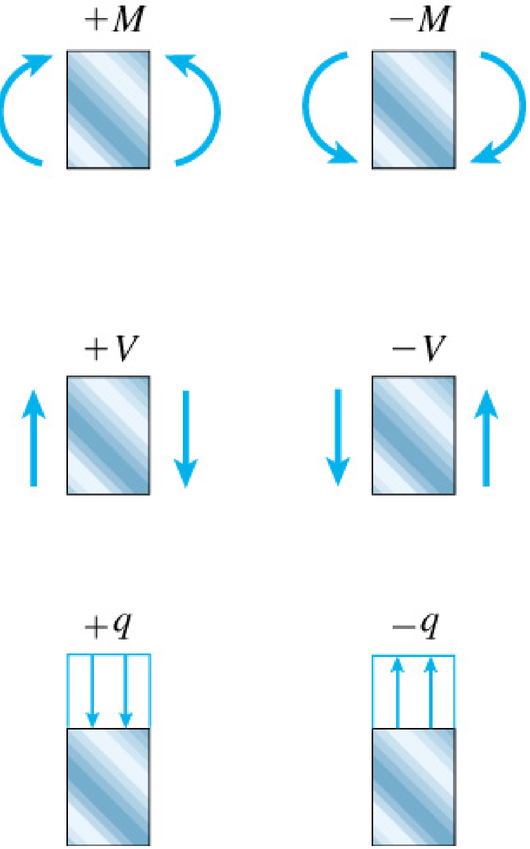
The x and y axes are positive to the right and upwards, respectively.

The deflection v is positive upwards.

The slope $\delta v/\delta x$ and angle of rotation θ are positive when CCW with respect to the positive x axis.

The curvature κ is positive when the beam is bent concave upward

The bending moment M is positive when it produces compression in the upper part of the beam.



Additional equations can be obtained from the relations between bending moment ***M***, shear force ***V***, and intensity ***q*** of distributed load

$$\frac{\delta^2 \upsilon}{\delta x^2} = \frac{M}{EI} \qquad \frac{\delta V}{\delta x} = -q \qquad \frac{\delta M}{\delta x} = V$$

We will consider two different cases:

- Non prismatic beams (***EI*** changes with ***x***)
- Prismatic beams (***EI*** constant)

Nonprismatic Beams

$$\frac{\delta}{\delta x}\left(EI \frac{\delta^2 \upsilon}{\delta x^2}\right) = \frac{\delta M}{\delta x} = V$$

$$\frac{\delta^2}{\delta x^2}\left(EI \frac{\delta^2 \upsilon}{\delta x^2}\right) = \frac{\delta^2 M}{\delta x^2} = \frac{\delta V}{\delta x} = -q$$

Prismatic Beams

These equations will be referred to as the **bending-moment equation**, the **shear force equation** and the **load equation**, respectively.

$$EI\frac{\delta^3\upsilon}{\delta x^3}=\frac{\delta M}{\delta x}=V$$

$$EI\frac{\delta^4\upsilon}{\delta x^4}=\frac{\delta^2 M}{\delta x^2}=\frac{\delta V}{\delta x}=-q$$

Deflections by Integration of the Bending-Moment Equation

Regardless of the number of bending-moment expressions, the general procedure for solving the differential equations is as follows:

1. For each region of the beam we substitute the expression for M into the differential equation and integrate to obtain the slope $v' = \delta v/\delta x$.
2. Each such integration produces one constant of integration.
3. Next, we integrate the slope equation to obtain the corresponding deflection v.

3. Again each integration produces a new constant.
4. The two constants of integration for each region of the beam are evaluated from known conditions pertaining to the slopes and deflections.

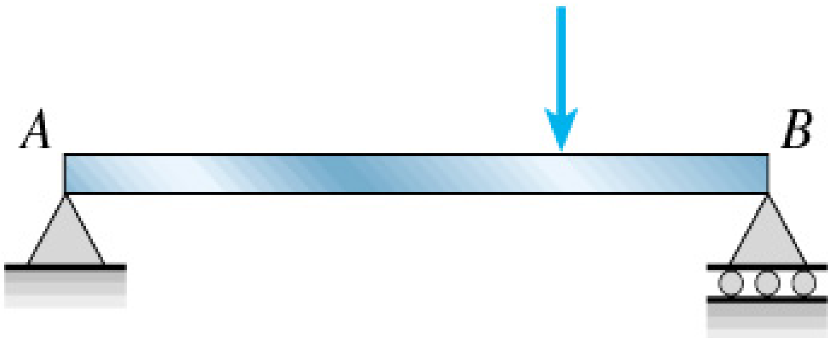
The conditions fall into three categories:

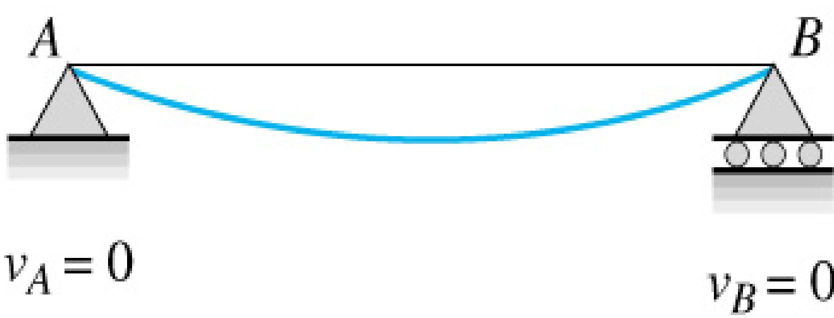
(1) Boundary Conditions

(2) Continuity Conditions

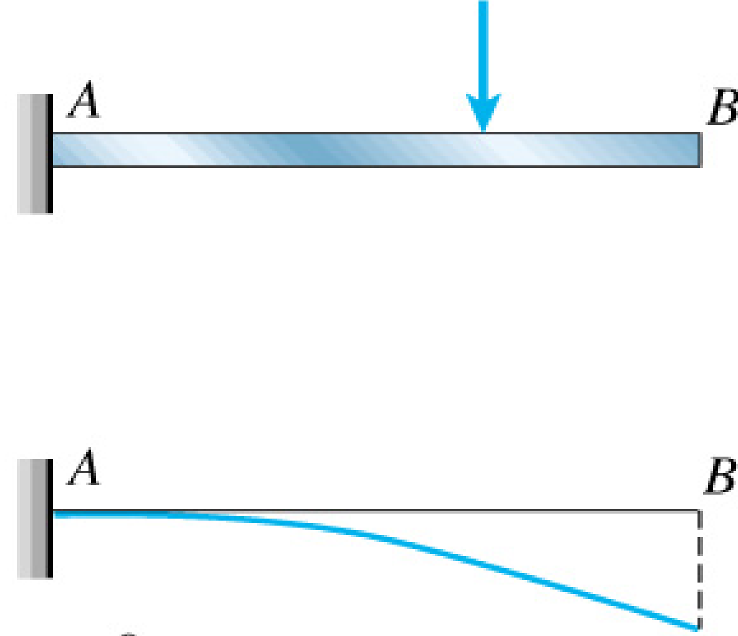
(3) Symmetry Conditions

1. **Boundary conditions**: Pertain to the deflections and slopes at the supports of a beam. Eg. At a simple support (either pin or roller) the deflection is zero and at a fixed support both the deflection and the slope are zero.



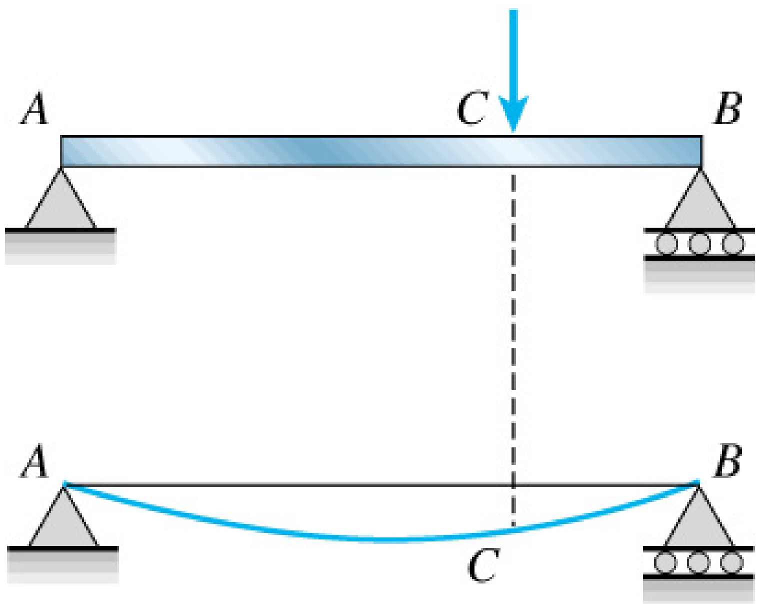


2. **Continuity conditions**: They occur at points where the regions of integration meet, (point ***C*** in the beam shown below). The deflection curve for this beam is physically continuous at point ***C***. Therefore the deflection of point ***C*** as determined for the left and right hand part of the beam must be equal.



$v_A = 0$

$v'_A = 0$

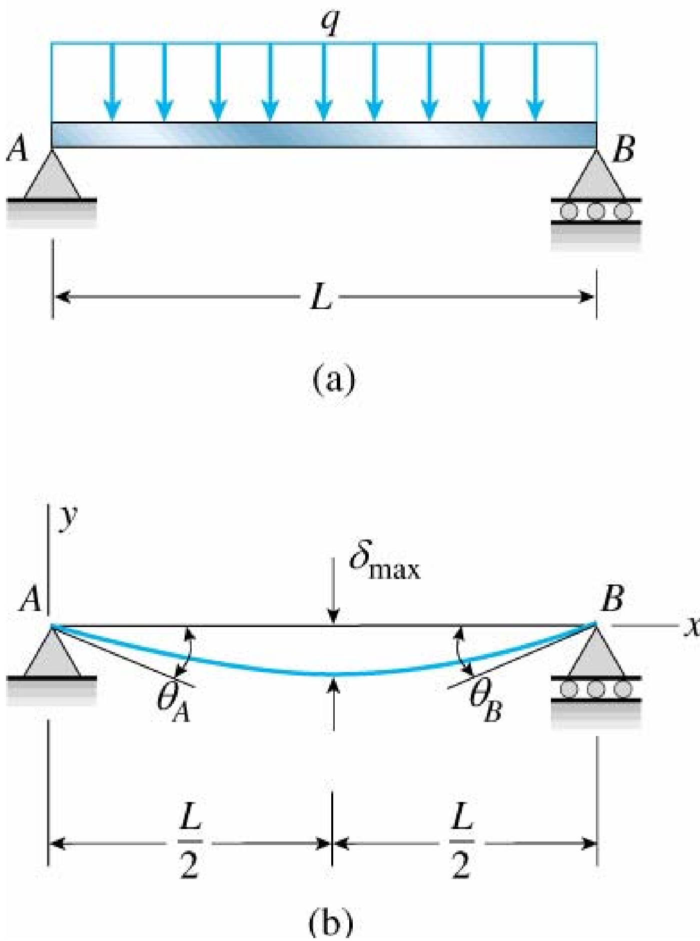


Similarly, the slopes found for each part of the beam must be equal at point ***C***.

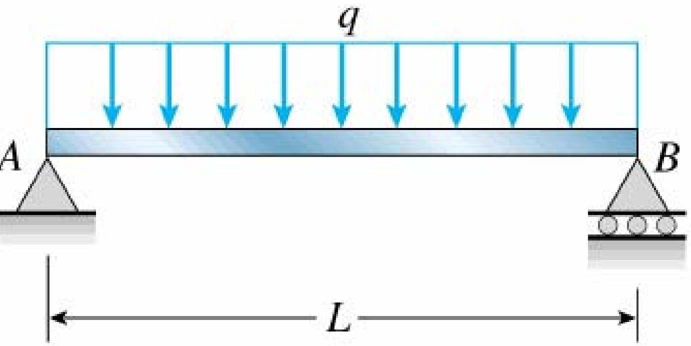
At point C: $(v)_{AC} = (v)_{CB}$

$(v')_{AC} = (v')_{CB}$

3. **Symmetry conditions:** They may also be available. Eg. If a simple beam supports a uniform load throughout its length, we know in advance that the slope of the deflection curve at the mid-point must be zero.



Each boundary, continuity and symmetry condition leads to an equation containing one or more of the constants of integration. The number of *independent* conditions always matches the number of constants of integration, we can always solve these equations for the constants. This method is sometimes referred to as the **method of successive integrations**.



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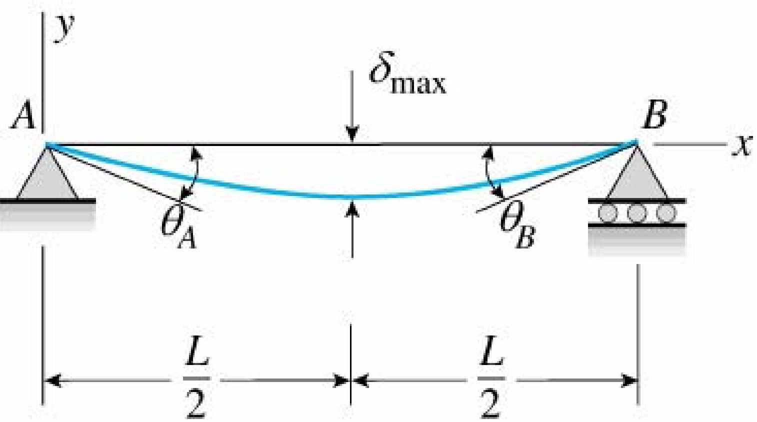


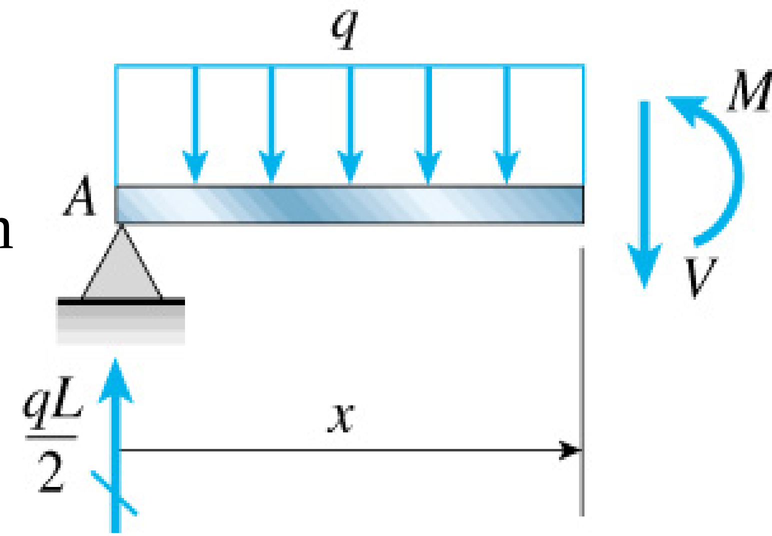
(b)

Example

Determine the equation of a deflection curve for a simple beam ***AB*** supporting a uniform load of intensity ***q*** acting throughout the span of the beam, as shown in the figure. Also, determine the maximum deflection δ_{max} at the midpoint of the beam and the angles of rotation θ_A and θ_B at the supports. (Note: the beam has length ***L*** and constant flexural rigidity ***EI***).

Solution

Bending moment in the beam. The bending moment at the cross section distance ***x*** from the left-hand support is obtained from the free-body diagram.



$$M = \frac{1}{2}qL(x) - qx\left(\frac{x}{2}\right) = EI\,\frac{\delta^2 \upsilon}{\delta x^2}$$

This equation can now be integrated to obtain the slope of the beam.

Slope of the beam:

$$EI\int \upsilon'' \delta x = \frac{1}{2}qL\int x\delta x - \frac{1}{2}q\int x^2 \delta x$$

C_1 is a constant of integration

$$EI\upsilon' = qL\frac{x^2}{4} - \frac{qx^3}{6} + C_1$$

Symmetry Condition: the slope of the deflection curve at midspan is equal to zero.

$$EI\upsilon'_{L/2} = qL\frac{\left(L/2\right)^2}{4} - \frac{q\left(L/2\right)^3}{6} + C_1 = 0 \qquad C_1 = -\frac{qL^3}{24}$$

$$\upsilon' = -\frac{q}{24EI}\left(L^3 - 6Lx^2 + 4x^3\right)$$

As expected, the slope is negative (i.e. clockwise) at the left-hand end of the beam (***x=0***), positive(i.e. anticlockwise) at the right-hand end (***x=L***), and equal to zero at the midpoint (***x = ½ L***).

Deflection of the beam : The deflection is obtained by integrating the equation for the slope.

$$EI\upsilon = qL\frac{x^3}{12} - \frac{qx^4}{24} - qL^3\frac{x}{24} + C_2$$

Boundary Condition: the deflection of the beam at the left-hand support is equal to zero (***ν* = *0*** when ***x* = *0***) or ***ν(0) = 0***

Applying this condition yields ***C₂=0***; hence the equation of the deflection curve is

$$\upsilon = -\left(\frac{qx}{24EI}\right)\left(L^3 - 2Lx^2 + x^3\right)$$

Maximum deflection :From symmetry we know that the maximum deflection occurs at the midpoint of the span. Thus, for ***x = L/2*** we obtain

$$\upsilon_{L/2} = -\frac{5qL^4}{384EI}$$ Downward deflection.

Angles of Rotation

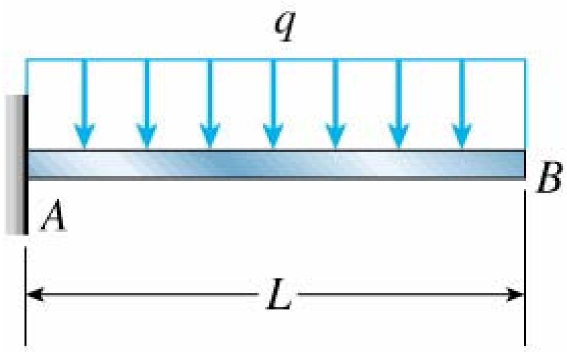
The maximum angles of rotation occur at the supports of the beam. Then for *x=0* and for *x=L*

$$\theta_A = -\upsilon'_{(x=0)} = \frac{qL^3}{24EI} \qquad \text{cw}$$

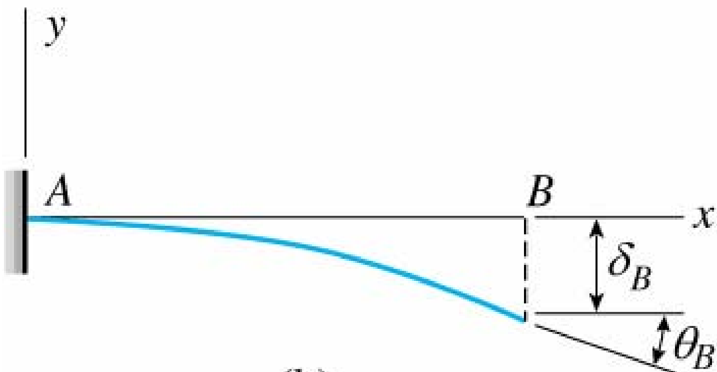
$$\theta_B = \upsilon'_{(x=L)} = \frac{qL^3}{24EI} \qquad \text{ccw}$$

Example

Determine the equation of the deflection curve for a cantilever beam ***AB*** subjected to a uniform load of intensity ***q*** (see figure). Also, determine the angle of rotation θ_B and the deflection δ_B at the free end. (The beam has a length ***L*** and a constant flexural rigidity ***EI***).



(a)



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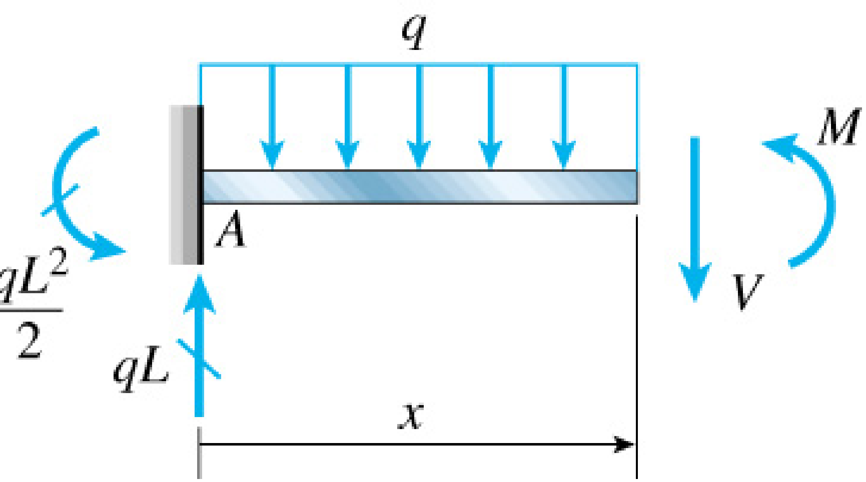
Solution



Bending moment equation

$$M = EI\upsilon'' = -\frac{qL^2}{2} + qLx - \frac{qx^2}{2}$$

Equation of the slope of the beam

$$EI\upsilon' = -\frac{qL^2 x}{2} + \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1$$

Boundary Condition: slope of the beam is zero at the support ***v'(0) = 0*** , then ***C₁ = 0***

$$EI\upsilon' = -\frac{qL^2x}{2} + \frac{qLx^2}{2} - \frac{qx^3}{6}$$

$$\upsilon' = -\frac{qx}{6EI}\left(3L^2 - 3Lx + x^2\right)$$

Deflection of the Beam:
Integration of the slope

$$EI\upsilon = -\frac{qL^2x^2}{4} + \frac{qLx^3}{6} - \frac{qx^4}{24} + C_2$$

$$boundary\_conditions \quad \upsilon_{x=0} = 0 \text{ then } C_2 = 0$$

$$\upsilon = -\frac{qx^2}{24EI}\left(6L^2 - 4Lx + x^2\right)$$

Angle of rotation and deflection at the free-end of the beam:

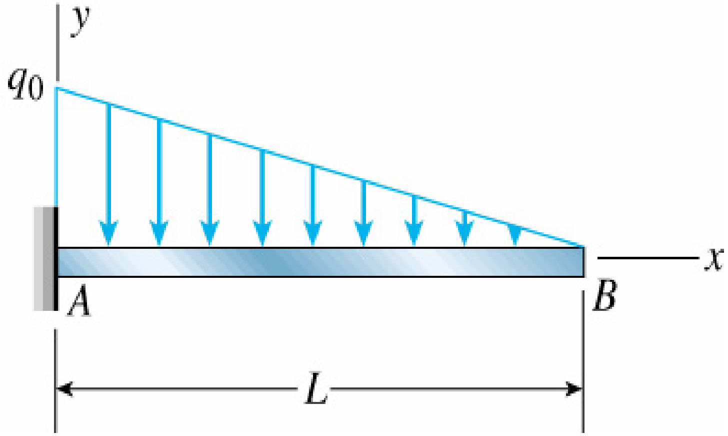
$$\theta_B = -\upsilon'_{x=L} = \frac{qL^3}{6EI} \qquad \delta_B = -\upsilon_{x=L} = \frac{qL^4}{8EI}$$

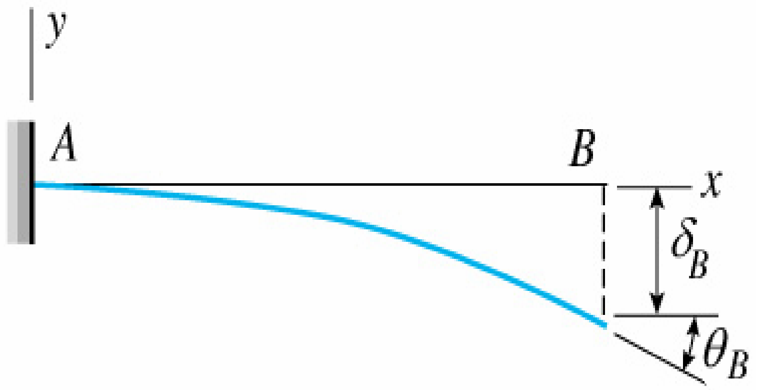
Deflections by Integration of the Shear-Force and Load Equations

The equations $EIv''' = V$ and $EIv'''' = -q$ in terms of the shear load V and the distributed load q may also be integrated to obtain the slopes and deflections.

Example

Determine the equation of the deflection curve for a cantilever beam AB supporting a triangularly distributed load of maximum intensity q_o (see figure below). (Note: the beam has length L and constant flexural rigidity EI)





Solution:

Differential equation of the deflection curve:

The intensity of the distributed load is given by the following equation:

$$q = \frac{q_o(L-x)}{L}$$

The fourth - order differential equation becomes

$$EI\nu'''' = -q = -\frac{q_o(L-x)}{L} \qquad V = EI\nu''' = \frac{q_o(L-x)^2}{2L} + C_1$$

Shear Force in the beam:

The first integration of the above equation gives

Because the shear force is zero at $\boldsymbol{x = L}$ then $\boldsymbol{\nu'''(L) = 0}$ and $\boldsymbol{C_1 = 0}$

The equation simplifies to :

$$V = EI\nu''' = \frac{q_o(L-x)^2}{2L}$$

Bending Moment in The Beam:

$$M = EIv'' = -\frac{q_o(L-x)^3}{6L} + C_2$$

Integrating a second time:

The bending moment is zero at the free end of the beam ***v''(L) = 0***

Therefore $\boldsymbol{C_2 = 0}$ and the equation simplifies to

$$M = EIv'' = -\frac{q_o(L-x)^3}{6L}$$

Slope and Deflection of the Beam:

The third and fourth integration yield

$$EIv' = \frac{q_o}{24L}(L-x)^4 + C_3 \qquad EIv = -\frac{q_o}{120L}(L-x)^5 + C_3 x + C_4$$

The boundary conditions at the fixed support, where the slope and the deflection equal zero, are ***v'(0) = 0*** ; ***v(0) = 0***, therefore

$$C_3 = -\frac{q_o L^3}{24} \qquad C_4 = \frac{q_o L^4}{120}$$

$$v' = -\frac{q_o x}{24LEI}(4L^3 - 6L^2 x + 4Lx^2 - x^3)$$

$$v = -\frac{q_o x^2}{120LEI}(10L^3 - 10L^2 + 5Lx^2 - x^3)$$

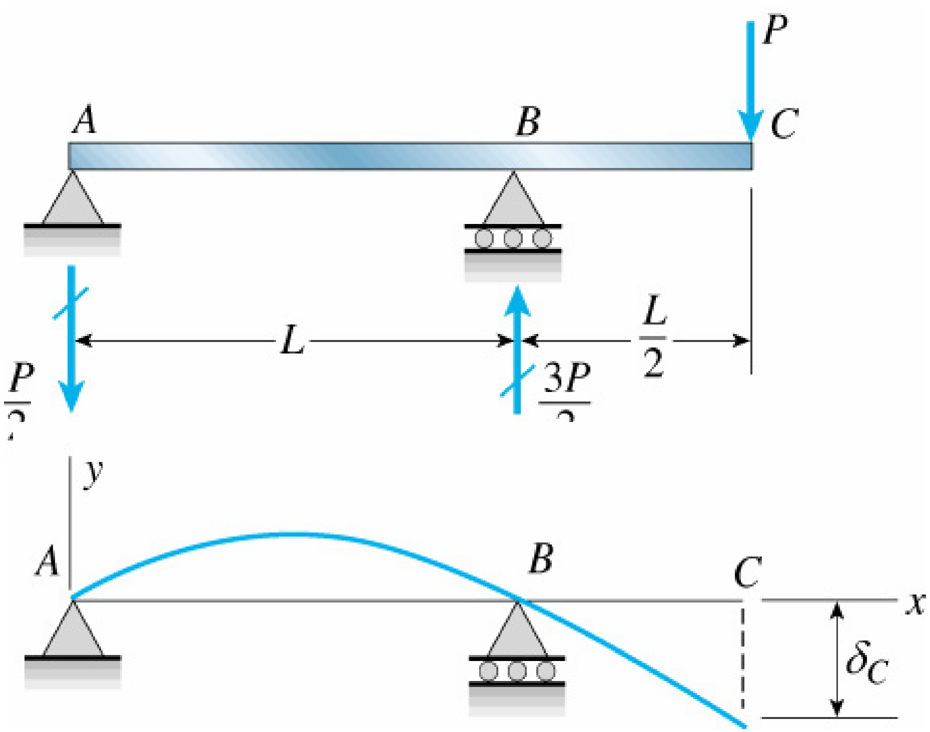
Angle of Rotation and Deflection at the Free end of the Beam:

The angle of rotation $\boldsymbol{\theta_B}$ and the deflection $\boldsymbol{\delta_B}$ at the free end of the beam are obtained by substituting $\boldsymbol{x = L}$ in the above equations:

$$\theta_B = -v'(L) = \frac{q_o L^3}{24EI} \qquad \delta_{\mathrm{B}} = -v(L) = \frac{q_o L^4}{30EI}$$

Example

A simple beam ***AB*** with an overhang ***BC*** supports a concentrated load ***P*** at the end of the overhang. The main span of the beam has a length ***L*** and the overhang has a length ½ ***L***.



Determine the equations of the deflection curve and the deflection δ_C at the end of the overhang (see figure). (Note: the beam has constant flexural rigidity ***EI***)

Solution :

We must write separate differential equations for parts ***AB*** and ***BC*** of the beam.

Reaction at support ***A = ½ P*** Reaction at support ***B = 3P / 2***

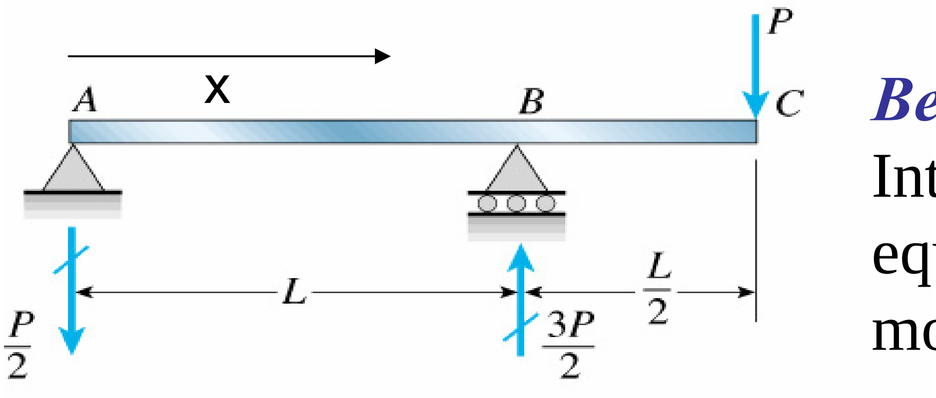
The shear forces in parts ***AB*** and ***BC*** are

$$V = -\tfrac{1}{2}P \quad (0 \leq x \leq L) \qquad\qquad V = P \quad (L \leq x \leq 3L/2)$$

In which ***x*** is measured from end ***A*** of the beam.

The third-order differential equations for the beam now become

$$EIv''' = V = -\tfrac{1}{2}P \quad (0 \leq x \leq L) \qquad EIv''' = V = P \quad (L \leq x \leq 3L/2)$$



Bending Moments of the Beam:

Integration of the preceding two equations yields the bending-moment equations:

$$M = EIv'' = -\tfrac{1}{2}Px + C_1 \qquad (0 \leq x \leq L)$$

$$M = EIv'' = Px + C_2 \qquad (L \leq x \leq 3L/2)$$

The bending moments at points A and C are zero

$$v''(0) = 0 \qquad\qquad v''(3L/2) = 0$$

Using these conditions we get $C_1 = 0$ and $C_2 = -3PL / 2$ therefore

$$M = EIv'' = -\tfrac{1}{2}Px \qquad (0 \le x \le L)$$

$$M = EIv'' = -\tfrac{1}{2}P(3L - 2x) \qquad (L \le x \le 3L/2)$$

These equations can be verified by determining the bending moments from free-body diagrams and equations of equilibrium.

Slopes and Deflections of the Beam

The next integration yield the slopes:

$$EIv' = -Px^2 / 4 + C_3 \qquad (0 \le x \le L)$$

$$EIv' = -\tfrac{1}{2}Px(3L - x) + C_4 \qquad (L \le x \le 3L/2)$$

The only condition on the slopes is the continuity condition at point ***B*** (slope as found for part ***AB = slope as found for part BC***)

$$-Px^2 / 4 + C_3 = -\tfrac{1}{2}Px(3L - x) + C_4 \quad \text{for } x = L \qquad \text{then}$$

$$C_4 = C_3 + \tfrac{3}{4}PL^2$$

The third and last integration give

$$EI\nu = -(1/12)Px^3 + C_3x + C_5 \qquad (0 \le x \le L)$$

$$EI\nu = -(1/12)Px^2\,(9L - 2x) + C_4x + C_6 \qquad (L \le x \le 3L/2)$$

The boundary conditions are that the deflections in ***A*** and ***B*** are zero.

$\nu(0)=0$; $\nu(L)=0$ Hence, we obtain: $C_5=0$; $C_3=PL^2/12$ and substituting in previous equations $C_4 = 5PL^2/6$ and $C_6 = -PL^3/4$

All constants of integrations have now been evaluated. Substituting in the above equations

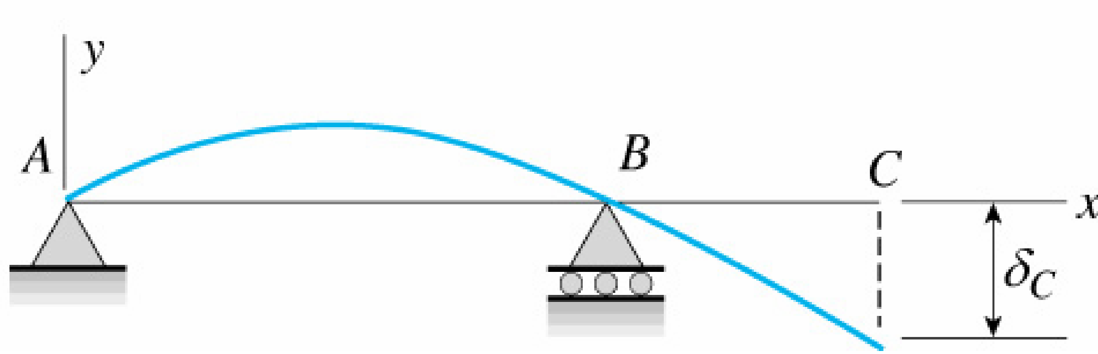
$$\nu = (1/12EI)\,Px\,(L^2 - x^2) \qquad (0 \le x \le L)$$

$$\nu = -(1/12EI)\,P\,(3L^3 - 10L^2x + 9Lx^2 - 2x^3) \qquad (L \le x \le 3L/2)$$

Note: the deflection is always positive (upward) in part ***AB*** and negative (downward) in ***BC***.

Deflection at the end of the overhang: $x = 3L/2$

$$\delta_C = -\nu_{(3L/2)} = PL^3/(8EI)$$

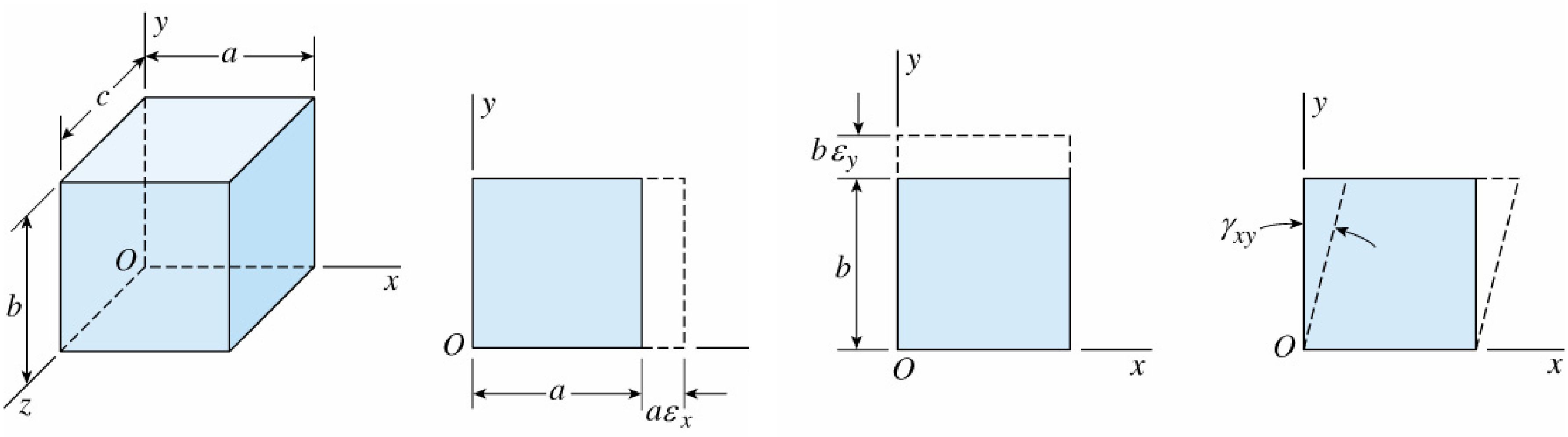


Plain Strain

We will derive the transformation equations that relate the strains in inclined directions to the strain in the reference directions.

State of plain strain - the only deformations are those in the ***xy*** plane, i.e. it has only three strain components ε_x, ε_y and γ_{xy}.

Plain stress is analogous to plane stress, but under ordinary conditions they do not occur simultaneously, except when $\sigma_x = -\sigma_y$ and when $\nu = 0$



Strain components ε_x, ε_y, and γ_{xy} in the ***xy*** plane (plane strain).

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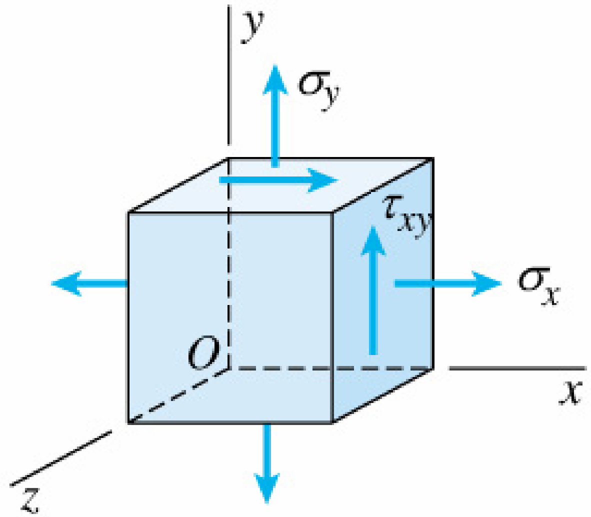
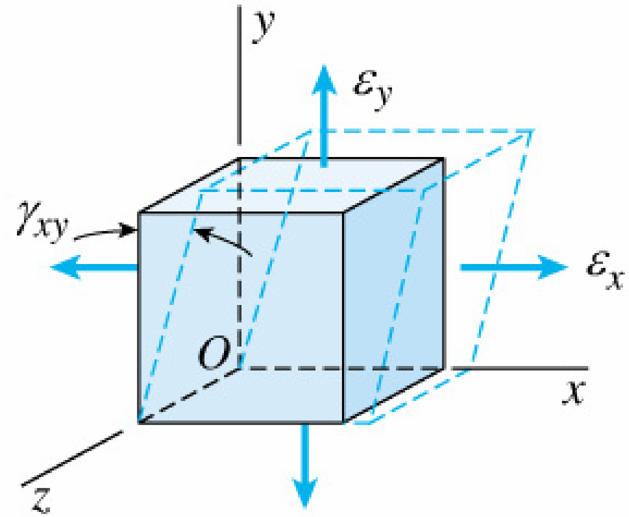
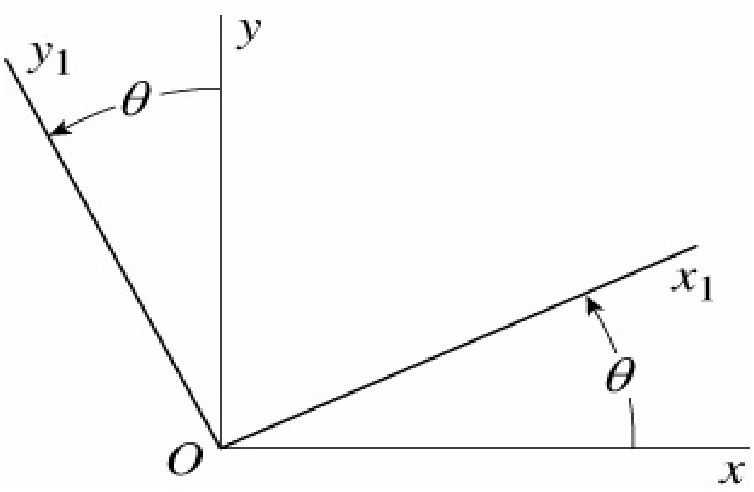
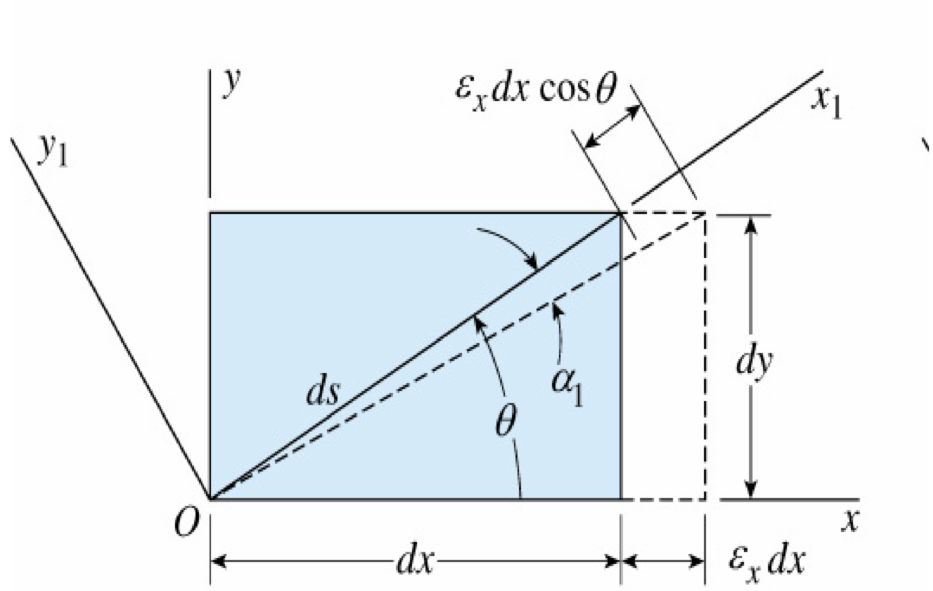
<tr><th></th><th>Plane stress</th><th>Plane strain</th></tr>
<tr><td></td><td>
</td><td>
</td></tr>
<tr><td>Stresses</td><td>$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$
σ_x, σ_y, and τ_{xy} may have nonzero values</td><td>$\tau_{xz} = 0$ $\tau_{yz} = 0$
σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values</td></tr>
<tr><td>Strains</td><td>$\gamma_{xz} = 0$ $\gamma_{yz} = 0$
ε_x, ε_y, ε_z, and γ_{xy} may have nonzero values</td><td>$\varepsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$
ε_x, ε_y, and γ_{xy} may have nonzero values</td></tr>
</table>

Comparison of plane stress and plane strain.

Transformation Equations for Plain Strain

Assume that the strain ε_x, ε_y and γ_{xy} associated with the ***xy*** plane are known. We need to determine the normal and shear strains (ε_{x1} and γ_{x1y1}) associated with the x_1y_1 axis. ε_{y1} can be obtained from the equation of ε_{x1} by substituting $\theta + 90$ for θ.

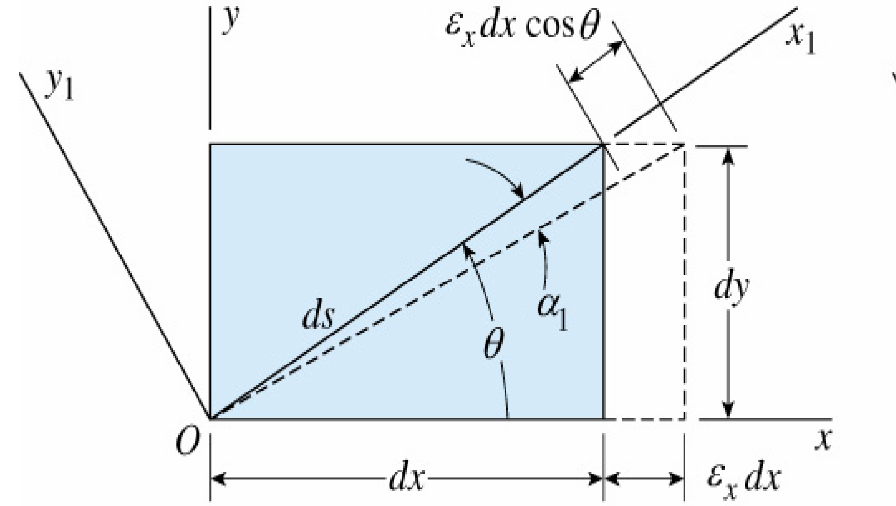


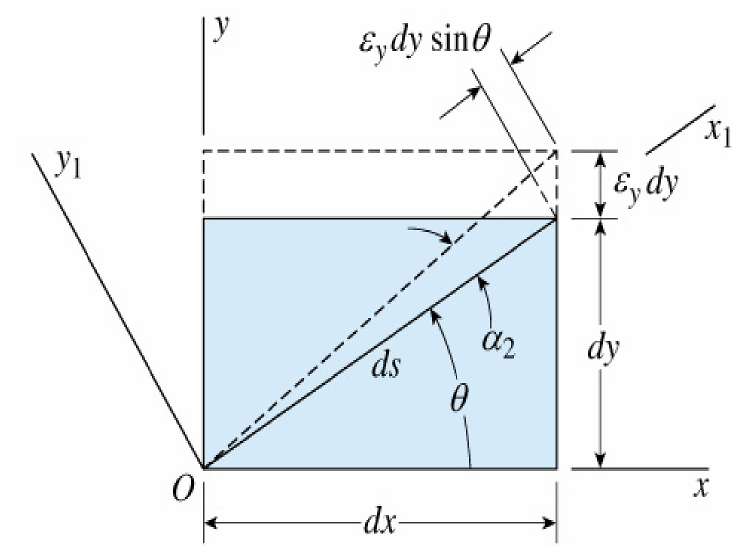


For an element of size $\delta x\ \delta y$

In the $\boldsymbol{x}$ direction, the strain ε_x produces an elongation $\varepsilon_x\ \delta x$.

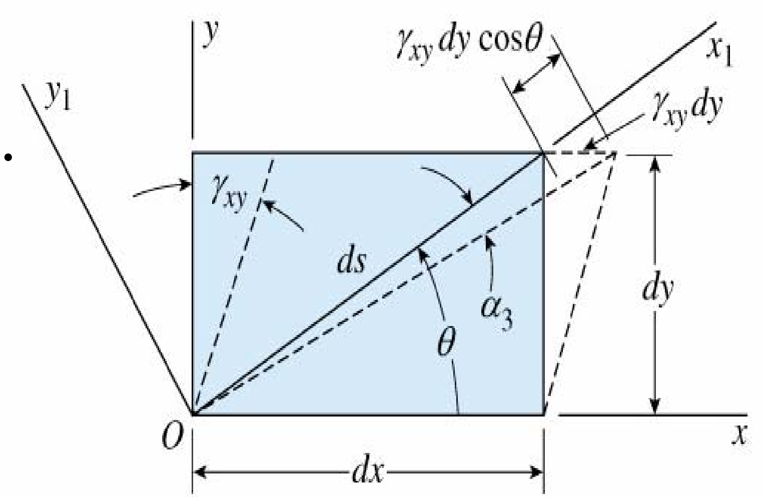
The diagonal increases in length by $\varepsilon_x\ \delta x \cos\theta$.





In the $\boldsymbol{y}$ direction, the strain ε_y produces an elongation $\varepsilon_y\ \delta y$. The diagonal increases in length by $\varepsilon_y\ \delta y \sin\theta$.

The shear strain γ_{xy} produces a distortion. The upper face moves $\gamma_{xy}\ \delta y$. This deformation results in an increase of the diagonal equal to: $\gamma_{xy}\ \delta y \cos\theta$



The total increase Δd of the diagonal is the sum of the preceding three expressions, thus:

$$\Delta d = \varepsilon_x \, \delta x \cos\theta + \varepsilon_y \, \delta y \sin\theta + \gamma_{xy} \, \delta y \cos\theta$$

The normal strain ε_{x1} in the x_1 direction is equal to the increase in length divided by the initial length δs of the diagonal.

$$\varepsilon_{x1} = \Delta d / ds = \varepsilon_x \cos\theta \, \delta x/\delta s + \varepsilon_y \sin\theta \, \delta y/\delta s + \gamma_{xy} \cos\theta \, \delta y/\delta s$$

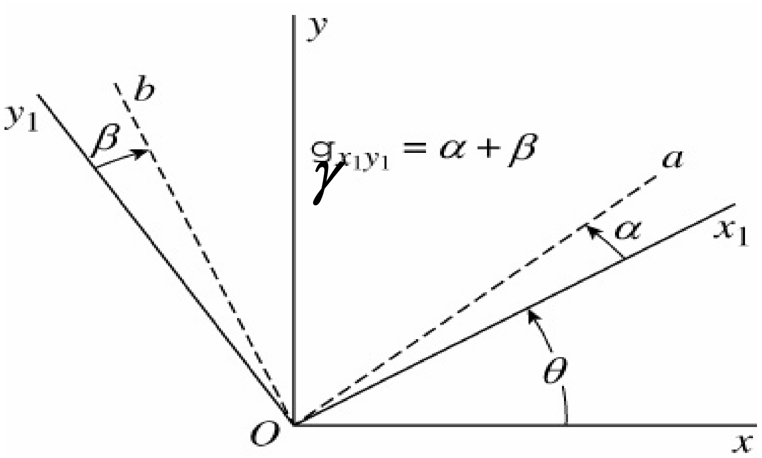
Observing that $\delta x/\delta s = \cos\theta$ and $\delta y/\delta s = \sin\theta$

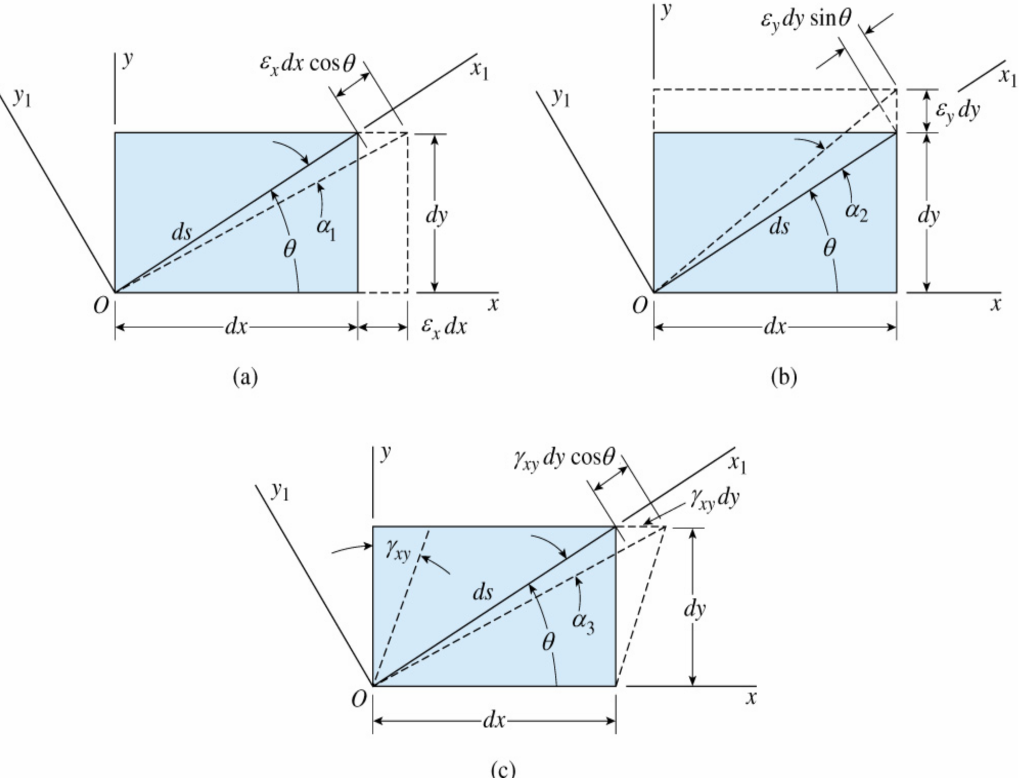
$$\varepsilon_{X1} = \varepsilon_X \cos^2\theta + \varepsilon_Y \sin^2\theta + \gamma_{XY} \sin\theta \cos\theta$$

$$\varepsilon_{X1} = \varepsilon_X \cos^2\theta + \varepsilon_Y \sin^2\theta + \left(\frac{\gamma_{XY}}{2}\right) 2 \sin\theta \cos\theta$$

Shear Strain γ_{x1y1} associated with x_1y_1 axes.

This strain is equal to the decrease in angle between lines in the material that were initially along the x_1 and y_1 axes. ***Oa*** and ***Ob*** were the lines initially along the x_1 and y_1 axis respectively. The deformation caused by the strains ε_x, ε_y and γ_{xy} caused the ***Oa*** and ***Ob*** lines to rotate and angle α and β from the x_1 and y_1 axis respectively. The shear strain γ_{x1y1} is the decrease in angle between the two lines that originally were at right angles, therefore, $\gamma_{x1y1} = \alpha+\beta$.





The angle α can be found from the deformations produced by the strains ε_x, ε_y and γ_{xy} . The strains ε_x and γ_{xy} produce a cw-rotation, while the strain ε_y produces a ccw-rotation.

Let us denote the angle of rotation produced by ε_x , ε_y and γ_{xy} as α_1 , α_2 and α_3 respectively. The angle α_1 is equal to the distance $\varepsilon_x\, \delta x \sin\theta$ divided by the length δs of the diagonal:

$\alpha_1 = \varepsilon_x \sin\theta\, dx/ds$ $\qquad$ $\alpha_2 = \varepsilon_y \cos\theta\, dy/ds$ $\qquad$ $\alpha_3 = \gamma_{xy} \sin\theta\, dy/ds$

Observing that $dx/ds = \cos\,\theta$ and $dy/ds = \sin\,\theta$. The resulting ccw-rotation of the diagonal is

$$\alpha = -\,\alpha_1 + \alpha_2 - \alpha_3 = -\,(\varepsilon_x - \varepsilon_y) \sin\theta \cos\theta\ -\ \gamma_{xy} \sin^2\theta$$

The rotation of line ***Ob*** which initially was at at ***90º*** to the line ***Oa*** can be found by substituting $\theta+90$ for θ in the expression for α.

Because β is positive when clockwise. Thus

$$\beta = (\varepsilon_x - \varepsilon_y)\sin(\theta + 90)\cos(\theta + 90) + \gamma_{xy}\sin^2(\theta+90)$$

$$\beta = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}\cos^2\theta$$

Adding α and β gives the shear strain γ_{x1y1}

$$\gamma_{x1y1} = \alpha + \beta = -2(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

To put the equation in a more useful form:

$$\frac{\gamma_{X1Y1}}{2} = -\varepsilon_X \sin\theta\cos\theta + \varepsilon_Y \sin\theta\cos\theta + \frac{\gamma_{XY}}{2}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\varepsilon_{X1} = \varepsilon_X \cos^2\theta + \varepsilon_Y \sin^2\theta + \left(\frac{\gamma_{XY}}{2}\right) 2\sin\theta\cos\theta$$

$$\varepsilon_{Y1} = \varepsilon_X \sin^2\theta + \varepsilon_Y \cos^2\theta - \left(\frac{\gamma_{XY}}{2}\right) 2\sin\theta\cos\theta$$

$$\begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \frac{\gamma_{X1Y1}}{2} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \left(\cos^2\theta - \sin^2\theta\right) \end{bmatrix} \begin{bmatrix} \varepsilon_{X} \\ \varepsilon_{Y} \\ \frac{\gamma_{XY}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \frac{\gamma_{X1Y1}}{2} \end{bmatrix} = [T] \times \begin{bmatrix} \varepsilon_{X} \\ \varepsilon_{Y} \\ \frac{\gamma_{XY}}{2} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \left(\cos^2\theta - \sin^2\theta\right) \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{X} \\ \varepsilon_{Y} \\ \frac{\gamma_{XY}}{2} \end{bmatrix} = [T]^{-1} \times \begin{bmatrix} \varepsilon_{X1} \\ \varepsilon_{Y1} \\ \frac{\gamma_{X1Y1}}{2} \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{zy} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = Strain\_Tensor$$

Transformation Equations for Plain Strain

Using known trigonometric identities, the transformation equations for plain strain becomes:

$$\varepsilon_{X1} = \frac{(\varepsilon_X + \varepsilon_Y)}{2} + \frac{(\varepsilon_X - \varepsilon_Y)}{2}\cos 2\theta + \frac{\gamma_{XY}}{2}\sin 2\theta$$

$$\frac{\gamma_{X1Y1}}{2} = -\frac{(\varepsilon_X - \varepsilon_Y)}{2}\sin 2\theta + \frac{\gamma_{XY}}{2}\cos 2\theta$$

These equations are counterpart of the equations for plane stress where ε_{x1}, ε_x, γ_{x1y1} and γ_{xy} correspond to σ_{x1}, σ_x, τ_{x1y1} and τ_{xy} respectively. There are also counterparts for principal stress and Mohr's circle. $\varepsilon_{x1} + \varepsilon_{y1} = \varepsilon_x + \varepsilon_y$

Principal Strains

The angle for the principal strains is

$$\tan 2\theta_P = \frac{\gamma_{XY}}{\varepsilon_X - \varepsilon_Y}$$

The value for the principal strains are

$$\varepsilon_1 = \frac{(\varepsilon_X + \varepsilon_Y)}{2} + \sqrt{\left(\frac{\varepsilon_X - \varepsilon_Y}{2}\right)^2 + \left(\frac{\gamma_{XY}}{2}\right)^2}$$

$$\varepsilon_2 = \frac{(\varepsilon_X + \varepsilon_Y)}{2} - \sqrt{\left(\frac{\varepsilon_X - \varepsilon_Y}{2}\right)^2 + \left(\frac{\gamma_{XY}}{2}\right)^2}$$

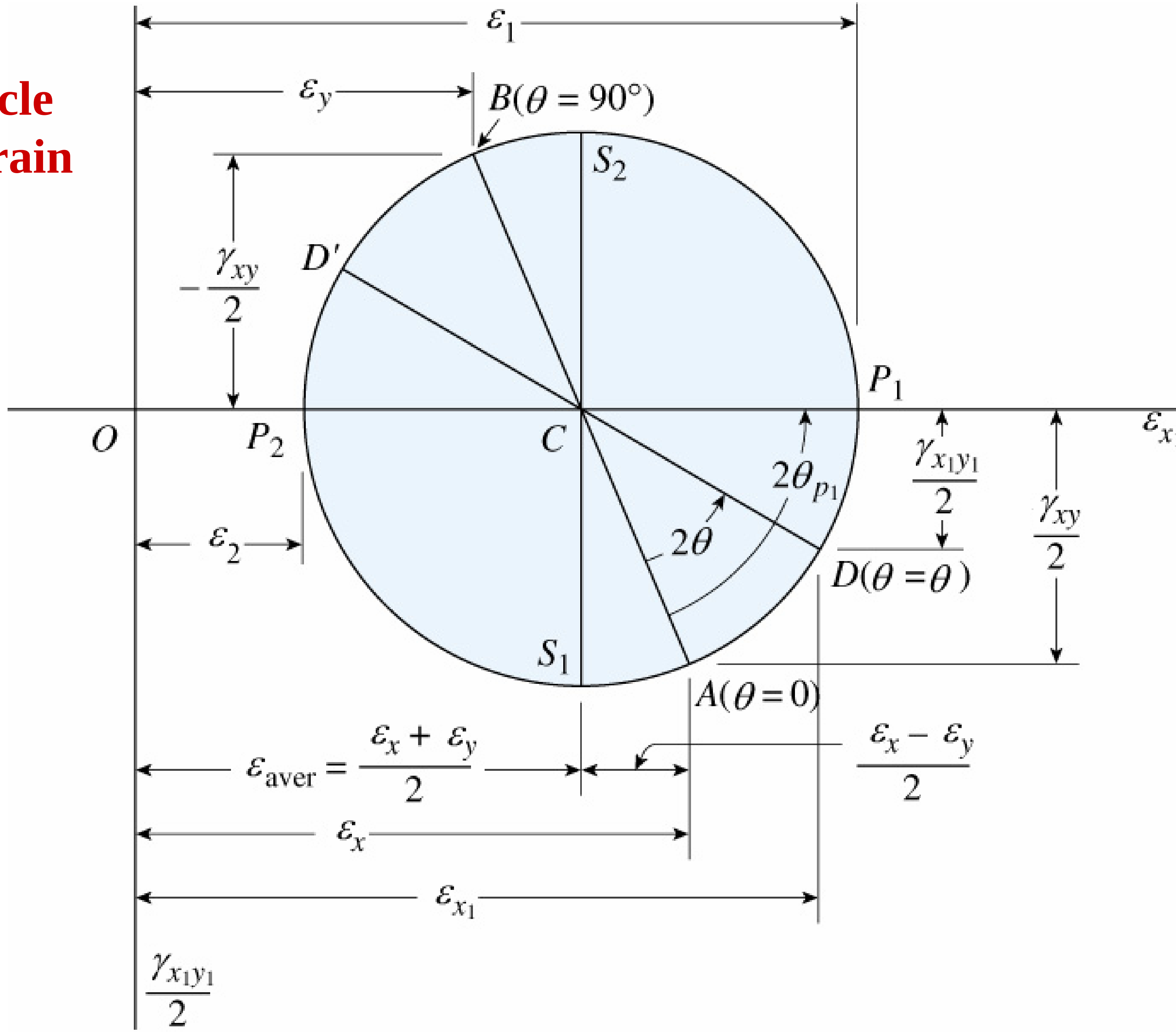
Maximum Shear

The maximum shear strains in the ***xy*** plane are associated with axes at ***45o*** to the directions of the principal strains:

$$\frac{\gamma_{MAX}}{2} = +\sqrt{\left(\frac{\varepsilon_X - \varepsilon_Y}{2}\right)^2 + \left(\frac{\gamma_{XY}}{2}\right)^2} \quad \text{or} \quad \gamma_{MAX} = (\varepsilon_1 - \varepsilon_2)$$

$$\frac{\gamma_{MAX}}{2} = \frac{(\varepsilon_1 - \varepsilon_2)}{2}$$

Mohr's Circle for Plane Strain



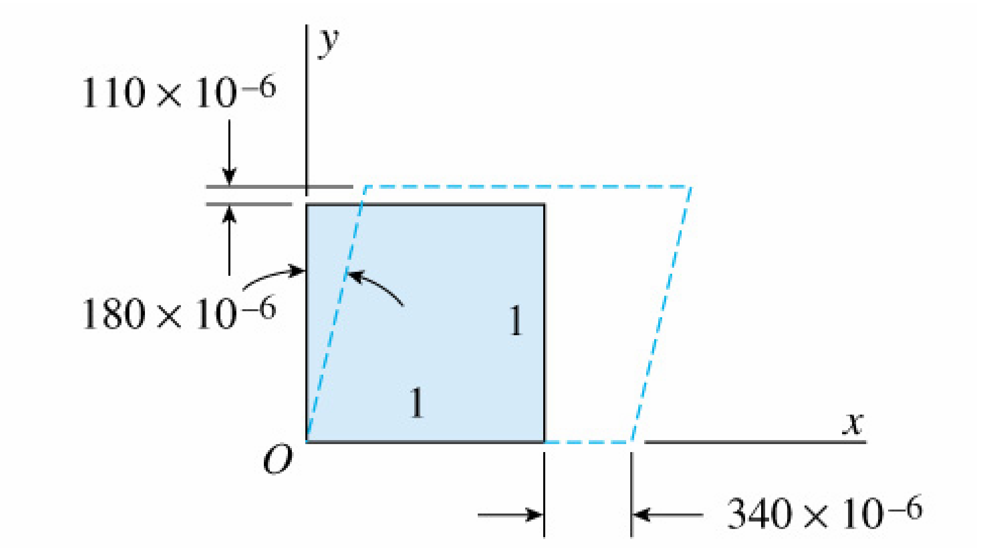
Example

An element of material in plane strain undergoes the following strains: $\varepsilon_x = 340x10^{-6}$

$\varepsilon_y = 110x10^{-6}$

$\gamma_{xy} = 180x10^{-6}$

Determine the following: (a) the strains of an element oriented at an angle $\theta = 30^o$; (b) the principal strains and (c) the maximum shear strains.



Solution

$$\varepsilon_{X1} = \frac{(\varepsilon_X + \varepsilon_Y)}{2} + \frac{(\varepsilon_X - \varepsilon_Y)}{2}\cos 2\theta + \frac{\gamma_{XY}}{2}\sin 2\theta$$

$$\frac{\gamma_{X1Y1}}{2} = -\frac{(\varepsilon_X - \varepsilon_Y)}{2}\sin 2\theta + \frac{\gamma_{XY}}{2}\cos 2\theta$$

Then

$\varepsilon_{x1} = 225\text{x}10^{-6} + (115\text{x}10^{-6}) \cos 60^o + (90\text{x}10^{-6}) \sin 60^o = 360\text{x}10^{-6}$

$\frac{1}{2} \gamma_{x1y1} = -(115\text{x}10^{-6}) (\sin 60^o) + (90\text{x}10^{-6})(\cos 60^o) = -55\text{x}10^{-6}$

Therefore $\gamma_{x1y1} = -110\text{x}10^{-6}$

The strain ε_{y1} can be obtained from the equation $\varepsilon_{x1} + \varepsilon_{y1} = \varepsilon_x + \varepsilon_y$

$\varepsilon_{y1} = (340 + 110 - 360)10^{-6} = 90\text{x}10^{-6}$

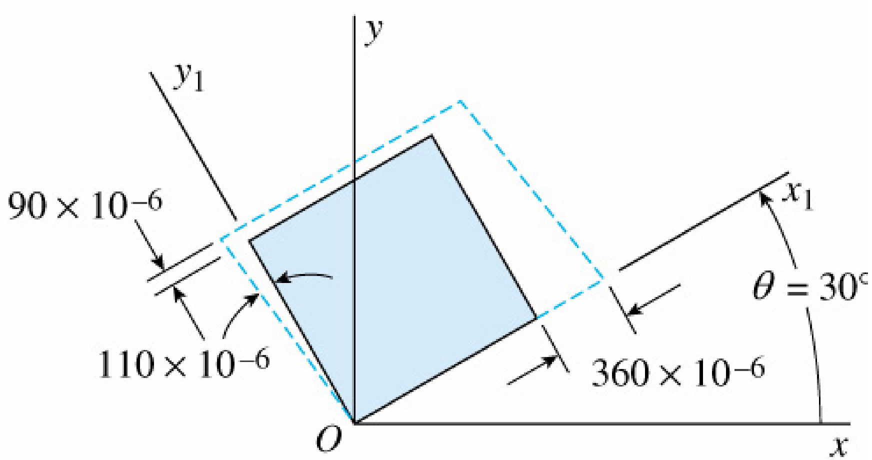
(b) ***Principal Strains***

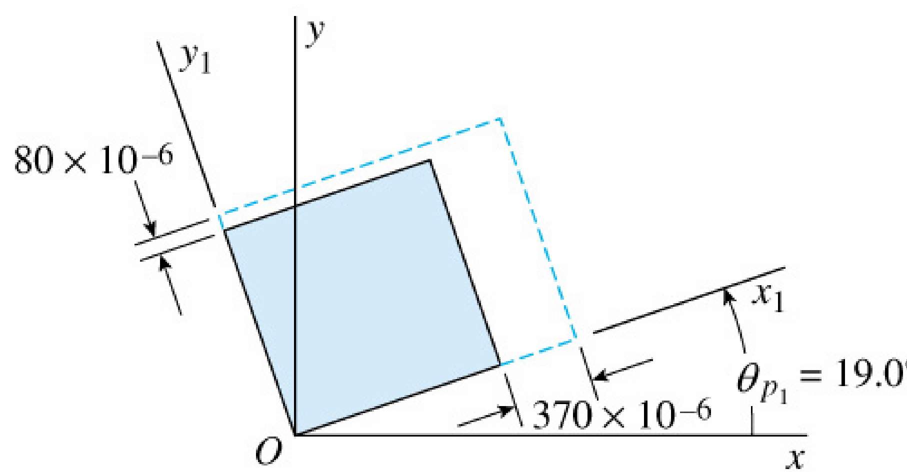
The principal strains are readily determine from the following equations:

$$\varepsilon_1 = \frac{(\varepsilon_X + \varepsilon_Y)}{2} + \sqrt{\left(\frac{\varepsilon_X - \varepsilon_Y}{2}\right)^2 + \left(\frac{\gamma_{XY}}{2}\right)^2}$$

$$\varepsilon_2 = \frac{(\varepsilon_X + \varepsilon_Y)}{2} - \sqrt{\left(\frac{\varepsilon_X - \varepsilon_Y}{2}\right)^2 + \left(\frac{\gamma_{XY}}{2}\right)^2}$$

$\varepsilon_1 = 370\text{x}10^{-6}$ $\varepsilon_2 = 80\text{x}10^{-6}$





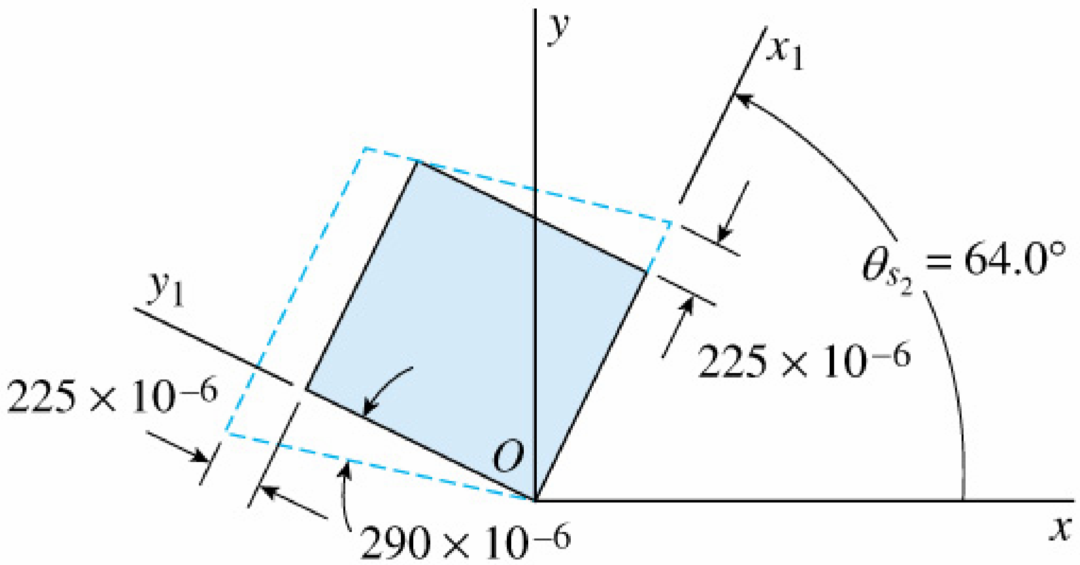
(c) **Maximum Shear Strain**

The maximum shear strain is calculated from the equation:

$\frac{1}{2}\gamma_{max} = SQR[((\varepsilon_x - \varepsilon_y)/2)^2 + (\frac{1}{2}\gamma_{xy})^2]$ or $\gamma_{max} = (\varepsilon_1 - \varepsilon_2)$

Then $\gamma_{max} = 290 \times 10^{-6}$

The normal strains of this element is $\varepsilon_{aver} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) = 225 \times 10^{-6}$



For 3-D problems

Tensor strain matrix from engineering strains

$$[\varepsilon]=\begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

Which is a symmetrical matrix. As in the case of stresses:

$$\begin{bmatrix} \varepsilon_x-\varepsilon & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y-\varepsilon & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z-\varepsilon \end{bmatrix}\begin{bmatrix} k \\ l \\ m \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} \varepsilon_x-\varepsilon & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y-\varepsilon & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z-\varepsilon \end{vmatrix}=0$$

Characteristic equation

$$\varepsilon_p^3 - I_1\varepsilon_p^2 + I_2\varepsilon_p - I_3 = 0$$

$$\varepsilon_1 \rangle \varepsilon_2 \rangle \varepsilon_3$$

Maximum shear strain

$$\frac{\gamma_{max}}{2} = \left| max\left(\frac{\varepsilon_1-\varepsilon_2}{2}, \frac{\varepsilon_2-\varepsilon_3}{2}, \frac{\varepsilon_3-\varepsilon_1}{2}\right)\right|$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} = \gamma_{xy}/2 & \varepsilon_{xz} = \gamma_{xz}/2 \\ \varepsilon_{yx} = \gamma_{yx}/2 & \varepsilon_{yy} & \varepsilon_{yz} = \gamma_{yz}/2 \\ \varepsilon_{zx} = \gamma_{zx}/2 & \varepsilon_{zy} = \gamma_{zy}/2 & \varepsilon_{zz} \end{bmatrix}$$

Strain invariants

$$I_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$I_2 = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

$$I_3 = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_1\varepsilon_2\varepsilon_3$$

$$\varepsilon^3 - I_1 \cdot \varepsilon^2 + I_2 \cdot \varepsilon - I_3 = 0$$

$$I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

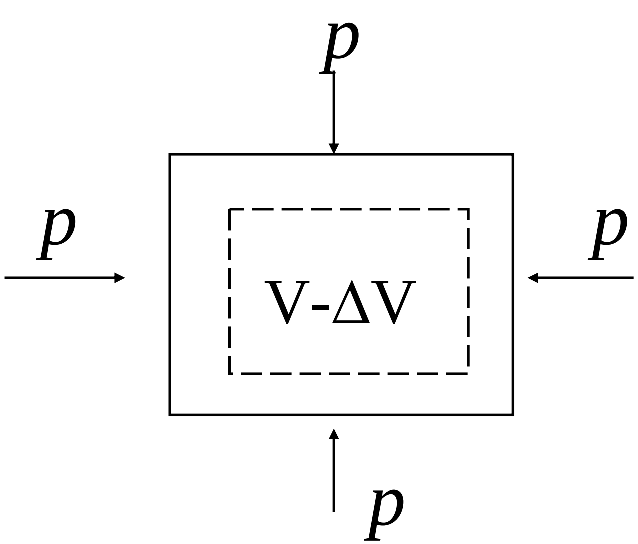
$$I_2 = \varepsilon_x \cdot \varepsilon_y + \varepsilon_y \cdot \varepsilon_z + \varepsilon_x \cdot \varepsilon_z - \left(\frac{\gamma_{xy}}{2}\right)^2 - \left(\frac{\gamma_{xz}}{2}\right)^2 - \left(\frac{\gamma_{yz}}{2}\right)^2$$

$$I_3 = \varepsilon_x \cdot \varepsilon_y \cdot \varepsilon_z + 2 \cdot \left(\frac{\gamma_{xy}}{2}\right) \cdot \left(\frac{\gamma_{xz}}{2}\right) \cdot \left(\frac{\gamma_{yz}}{2}\right) - \varepsilon_x \cdot \left(\frac{\gamma_{yz}}{2}\right)^2 - \varepsilon_y \cdot \left(\frac{\gamma_{xz}}{2}\right)^2 - \varepsilon_z \cdot \left(\frac{\gamma_{xy}}{2}\right)^2$$

Dilatation (Volume strain)

Under pressure: the volume will change

$$\Delta = \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$



Strain Deviator

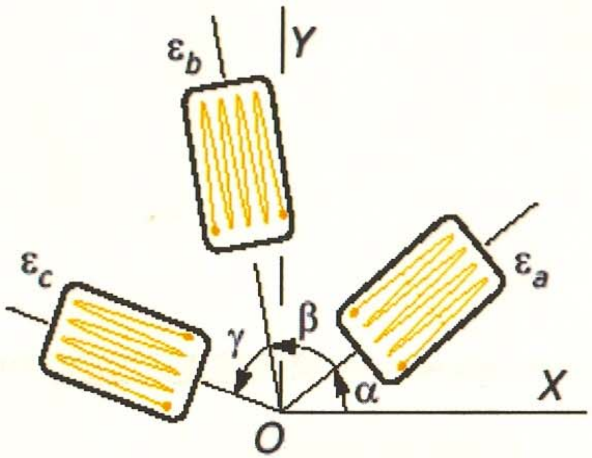
Mean strain $$\frac{\Delta}{3} = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3}$$

It produces a volume change (not a shape change)

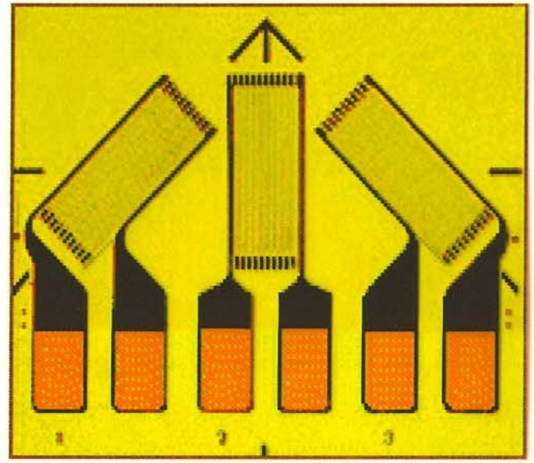
$$[D_\varepsilon] = \begin{bmatrix} \varepsilon_x - \frac{1}{3}\Delta & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y - \frac{1}{3}\Delta & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z - \frac{1}{3}\Delta \end{bmatrix}$$

Strain Deviator Matrix

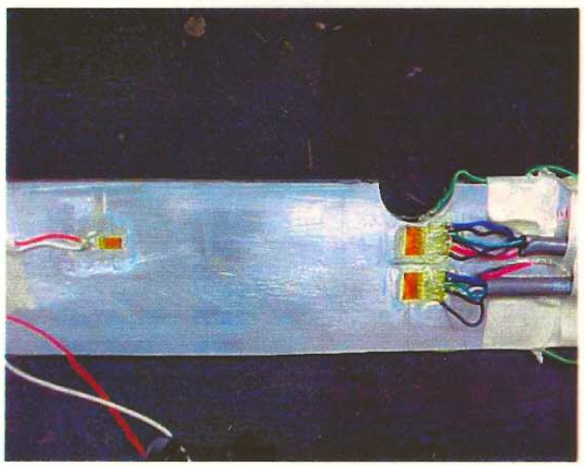
$$[D_\varepsilon] = \begin{bmatrix} \varepsilon_1 - \frac{1}{3}\Delta & 0 & 0 \\ 0 & \varepsilon_2 - \frac{1}{3}\Delta & 0 \\ 0 & 0 & \varepsilon_3 - \frac{1}{3}\Delta \end{bmatrix}$$



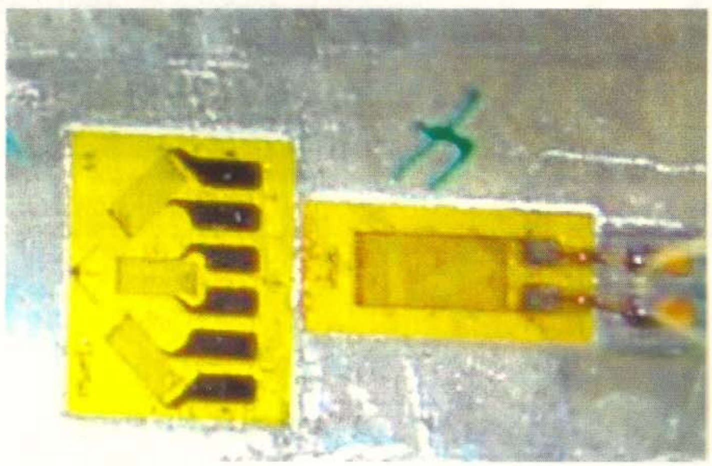
(a)



(b)



(c)



(d)

Application : Strain Gauge and Strain Rosette